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A Tutorial on Vectors and Attitude

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> *After Einstein had been awarded the Nobel prize and had written a popular book* Relativity; the Special and the General Theory*, a reporter came to Einstein's home in Berlin and asked one of the great physicist's nieces if she had read the book and understood it. "Oh, yes," she replied, "everything but the part on coordinate systems."*

We examine vectors and attitude in Engineering with greater attention to detail than in earlier works on the subject. This tutorial is an expansion of part of a survey [1] of attitude representations [1]. The most salient characteristic of this work is a greater emphasis on the distinction between physical vectors and their representation as column vectors. The need for such a treatment arises most strongly in attitude estimation, but logically, the need is present also in attitude dynamics and control.

Attitude is estimated by comparing *two different* column-vector representations of the *same* physical vector. Thus, from the beginning, the distinction between physical vectors and column vectors requires explicit treatment in attitude estimation. In works on attitude dynamics and control [2, 3, 4], however, this distinction is often overlooked or presented only in attitude statics, with the transition to attitude dynamics being made as much through verbal arguments as by mathematical development.

History plays a role in this situation. Attitude dynamics begins in the second half of the eighteenth century with the work of Euler. In contrast, the first journal article on attitude estimation did not appear until 1964 [5], an engineering note of only one-and-a-half pages, still cited today. More than a decade would pass beyond that event before the subject began to attract serious formal attention. The need to distinguish between physical vectors and their column-vector representations with respect to a basis had been recognized by Pars [6], who wrote in 1965

> "Strictly speaking we should distinguish between a vector **X** and the column matrix whose elements are its components, but in practice we shall often regard the terms vector and column matrix as synonymous, a usage which will not give rise to any confusion."

With the passage of half a century, including most of the space age, Pars' statement requires some emendation. In fact, failure to recognize the difference between physical vectors and column vectors has led sometimes to errors in spacecraft mission support software. Usually, these errors are corrected during software testing, but such errors should not occur in the first place. A contributing factor to such errors is the unfamiliarity of attitude work for many engineers, who generally receive little exposure to this area in university studies. As remarked in [7], attitude estimation and control "is generally considered the most complex and least intuitive of the space vehicle design disciplines." History plays a further role also in that attitude estimation, being a young subject, is not well represented as a research area at universities, although, as a practical activity, attitude determination occupies a large number of engineers in government and industry.

The reason that attitude dynamics texts could be casual about the distinction between physical vectors and their column-vector representations is that, once the Euler transport equation with the "$\boldsymbol{\omega}\times$" term is obtained, further development of the subject is on solid ground. Although the attitude matrix is the transformation matrix of column vectors from the inertial frame to the body frame, all dynamical calculations are generally carried out only in the body frame. Thus, the need for column vector representations with respect to both the body reference frame and the inertial reference frame largely disappears.

The present work treats physical vectors, column vectors, and the attitude only for attitude statics, where many misconceptions exist that affect the development of a more complete formalism embracing also attitude dynamics. The material here might appear to have been already well traveled, but that is not the case. Most treatments of vectors and attitude sidestep important conceptual issues that the present work meets head on. In addition, our goal is to be comprehensive rather than incremental.

The chief innovations of the present work compared to [1] include a more careful presentation of physical vectors and column vectors, the introduction of dual vectors, which greatly improves the presentation of dyadics, and a critical examination of the attitude dyadic. Many of the results that appear in [1] are derived here in a new way and examined more critically.

Physical Vectors and Column Vectors

We begin at the beginning. Vectors exist at three levels of abstraction. At the lowest level of abstraction are the *numerical column vectors*. If we write

$$\mathbf{v} = \begin{bmatrix} 1.0 \\ 2.3 \\ 8.32 \end{bmatrix} , \tag{1}$$

then $\mathbf{v}$ is a numerical column vector. The next level of abstraction is the *column-vector variable*, which we write in terms of components as

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} . \tag{2}$$

The difference between a numerical column vector and a column-vector variable is the same as that between writing "3" and writing "x." What makes both of these objects *column vectors* is that the two are 3×1 arrays, but what makes them vectors from the standpoint of Algebra is that column vectors are elements of a structure, a vector space, which has certain algebraic properties, presented below. Both numerical column vectors and column-vector variables are column vectors. For obvious reasons, numerical column vectors do not receive special attention in this work.

The third type of vector is a *physical vector*, described not by *components*, that is, not by three entries in an array, as in (1)–(2), but by some physical property. The physical description of the direction of a given star as seen from the Earth is meaningful by itself without specifying a reference frame for the calculation of components. A physical vector cannot be described by giving components (coordinates), because a physical vector is coordinate-free. The physical vector possesses a level of abstraction above that of the column-vector variable. We write a physical vector as "$\boldsymbol{v}$" (note the different font) together with a physical description or a relationship to other physical vectors. When we draw a vector as an arrow emanating from a point with an arrowhead at the other end, we are depicting a physical vector. A vector measurement, however, can have a numerical value, and therefore, must be a column vector. Both physical vectors and column vectors have an important place in Engineering, although this fact is acknowledged, perhaps, less frequently than should be the case.

Physical vectors, column-vector variables, and numerical column vectors, are the three vectorial quantities we must treat. The difficulty of that treatment arises from the fact that physical vectors and column vectors transform differently under rotation, that is, physical vectors and column vectors have different *tensorial* properties. In fact, vectors have algebraic properties as well as tensorial properties. The transformations are tantalizingly similar, but have subtle differences, and, if these subtle differences are not given proper attention, sign errors can result.

We use the Times bold italic font to denote a physical vector ($\boldsymbol{u}, \boldsymbol{v}, \hat{\boldsymbol{e}}_k, \ldots$) and the Helvetica bold font to denote column vectors ($\mathbf{u}, \mathbf{v}, \hat{\mathbf{e}}_k, \ldots$). For the most part, matrices are denoted by upper-case Helvetica letters ($\mathsf{A}, \mathsf{C}, \ldots$), and their entries by the corresponding upper-case Times italic letters ($A_{ij}, C_{ij}, \ldots$). In handwriting, we usually write a physical vector $\boldsymbol{u}$ as $\vec{u}$ and a column vector $\mathbf{u}$ as $\underline{u}$. A dyadic ($\mathcal{A}$) or vectrix ($\mathcal{V}$) is handwritten as $\overleftrightarrow{A}$ or $\overleftrightarrow{V}$.

Vector Spaces

Physical Vector Spaces

A physical *vector space* [8–10] consists of a set of physical vectors $\mathcal{V} = \{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}, \ldots\}$, a set of scalars $\mathcal{F} = \{a, b, c, \ldots\}$, and two operations, namely, physical vector addition and multiplication of a physical vector by a scalar. The set of scalars is a field, that is, a set whose elements have the same algebraic properties as the real numbers. In practical applications, the field is almost always the field of real numbers.

Note that none of the cited Mathematics books [8–10] makes the distinction between physical vectors and column vectors. In fact, all of these works illustrate the algebra of abstract vector spaces using column vectors.

The set $\mathcal{V}$ of physical vectors forms a group [10] under vector addition and satisfies necessarily the following conditions

(a) If $\boldsymbol{u}$ and $\boldsymbol{v}$ are physical vectors, then so is $\boldsymbol{u}+\boldsymbol{v}$.

(b) Addition of physical vectors is associative,

$$(\boldsymbol{u}+\boldsymbol{v})+\boldsymbol{w}=\boldsymbol{u}+(\boldsymbol{v}+\boldsymbol{w})\,, \tag{3}$$

and commutative,

$$\boldsymbol{u}+\boldsymbol{v}=\boldsymbol{v}+\boldsymbol{u}\,. \tag{4}$$

(c) There exists a physical vector $\boldsymbol{0}$ such that, for every physical vector $\boldsymbol{v}$

$$\boldsymbol{v}+\boldsymbol{0}=\boldsymbol{v}\,. \tag{5}$$

(d) For every physical vector $\boldsymbol{v}$, there exists a physical vector $-\boldsymbol{v}$ such that

$$\boldsymbol{v}+(-\boldsymbol{v})=\boldsymbol{0}\,. \tag{6}$$

By convention, we write

$$\boldsymbol{u}+(-\boldsymbol{v})\equiv\boldsymbol{u}-\boldsymbol{v} \tag{7}$$

and speak of vector subtraction.

Multiplication of a physical vector by a scalar is defined and satisfies

$$a\,(b\,\boldsymbol{v})=(ab)\,\boldsymbol{v}\,, \tag{8}$$

$$(a+b)\,\boldsymbol{v}=a\boldsymbol{v}+b\boldsymbol{v}\,, \tag{9}$$

$$a\,(\boldsymbol{u}+\boldsymbol{v})=a\boldsymbol{u}+a\boldsymbol{v}\,, \tag{10}$$

as well as

$$1\,\boldsymbol{v}=\boldsymbol{v}\,. \tag{11}$$

It follows that

$$0\,\boldsymbol{v}=\boldsymbol{0} \tag{12}$$

and

$$(-a)\,\boldsymbol{v}=-(a\,\boldsymbol{v})\,. \tag{13}$$

Equations (3)–(6) and (8)–(11) define the algebraic properties of physical vector spaces. Although (3)–(13) are written in terms of physical vectors, these equations must hold also for column vectors. The basic abstract algebraic description of a vector space above is, however, without physical content. In order to connect physical vectors to reality, we must find a way to connect physical vectors to the field $\mathcal{F}$. We use the symbol $\mathcal{V}$ to denote not only the set of physical vectors, but also the physical vector space.

The Scalar Product

We improve upon the physical vector space by adding a *scalar product* ($\cdot$), also called an *inner product* or a *'dot' product*, which allows us to associate with each pair of physical vectors a scalar. For physical vectors $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$, the scalar product $\boldsymbol{u} \cdot \boldsymbol{v}$ is an element of the field $\mathcal{F}$ and satisfies

$$\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{u}\,, \tag{14}$$

$$\boldsymbol{v} \cdot \boldsymbol{v} \geq 0\,, \tag{15}$$

$$\boldsymbol{v} \cdot \boldsymbol{v} = 0 \quad \text{if and only if} \quad \boldsymbol{v} = \boldsymbol{0}\,, \tag{16}$$

$$\boldsymbol{u} \cdot (a\boldsymbol{v}) = a(\boldsymbol{u} \cdot \boldsymbol{v})\,, \tag{17}$$

$$(\boldsymbol{u} + \boldsymbol{v}) \cdot \boldsymbol{w} = \boldsymbol{u} \cdot \boldsymbol{w} + \boldsymbol{v} \cdot \boldsymbol{w}\,. \tag{18}$$

A vector space with a scalar product is called an *inner-product space*.

The *magnitude* of a physical vector, written $|\boldsymbol{v}|$, is defined as

$$|\boldsymbol{v}| = (\boldsymbol{v} \cdot \boldsymbol{v})^{1/2}\,. \tag{19}$$

The scalar product for physical vectors, in order to have practical value, must be specified also physically, for example, as the product of the lengths of two physical vectors and the cosine of the included angle.

The Vector Product

In a three-dimensional physical vector space, a vector product ($\times$), also called a *'cross' product*, with values in $\mathcal{V}$, can be constructed satisfying

$$\boldsymbol{u} \times \boldsymbol{v} = -\boldsymbol{v} \times \boldsymbol{u}\,, \tag{20}$$

$$(a\boldsymbol{u}) \times \boldsymbol{v} = a(\boldsymbol{u} \times \boldsymbol{v})\,, \tag{21}$$

$$(\boldsymbol{u} + \boldsymbol{v}) \times \boldsymbol{w} = \boldsymbol{u} \times \boldsymbol{w} + \boldsymbol{v} \times \boldsymbol{w}\,. \tag{22}$$

The physical rule for the vector product is that $|\boldsymbol{u} \times \boldsymbol{v}| = |\boldsymbol{u}||\boldsymbol{v}| \sin\theta$, where θ is the angle needed to rotate $\boldsymbol{u}$ into $\boldsymbol{v}$ in their common plane, and the direction of $\boldsymbol{u} \times \boldsymbol{v}$ is given by the rule of the right hand screw.

Bases

Every physical vector $\boldsymbol{v}$ in three-dimensional space can be written in the form [8–10]

$$\boldsymbol{v} = a\,\boldsymbol{i} + b\boldsymbol{j} + c\,\boldsymbol{k}\,, \tag{23}$$

where $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ are the physical *basis vectors*, which may be any three linearly independent physical vectors in the physical vector space. The numbers a, b, and c are the *components* (or *coordinates*) of $\boldsymbol{v}$ with respect to this basis. As long as the physical basis vectors are linearly independent, the components exist and are unique..

A set of physical basis vectors, henceforth assumed to be linearly independent, and an origin constitute a reference frame. In this work, we assume there is only one origin and use ‘reference frame” to mean the same as “basis’ and“frame-independent” to mean the same as “basis-independent.”

If the physical basis vectors satisfy

$$\boldsymbol{i}\cdot\boldsymbol{j}=\boldsymbol{i}\cdot\boldsymbol{k}=\boldsymbol{j}\cdot\boldsymbol{k}=0\,, \tag{24}$$

the physical basis is called *orthogonal*. If, in addition,

$$\boldsymbol{i}\cdot\boldsymbol{i}=\boldsymbol{j}\cdot\boldsymbol{j}=\boldsymbol{k}\cdot\boldsymbol{k}=1\,, \tag{25}$$

the physical basis is called *orthonormal*, and the physical basis vectors are then written as $\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}}$. An orthonormal physical basis satisfying

$$\hat{\boldsymbol{\imath}}\times\hat{\boldsymbol{\jmath}}=\hat{\boldsymbol{k}}\,,\quad \hat{\boldsymbol{\jmath}}\times\hat{\boldsymbol{k}}=\hat{\boldsymbol{\imath}}\,,\quad \hat{\boldsymbol{k}}\times\hat{\boldsymbol{\imath}}=\hat{\boldsymbol{\jmath}}\,, \tag{26}$$

is further called *right-handed*.

Equations (26) of themselves do not specify physically the vector product. The rule is hidden in the choice of how we orient the three physical basis vectors $\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}}$ in space and the physical vector-product rule. Traditionally, we choose the physical coordinate axes as they are usually drawn and the physical vector-product rule to follow the right-handed screw rule. What we must keep in mind, from a geometric point of view, is that (26) do not make a set of physical coordinate axes *physically* right-handed, but rather we do by the choice of how we orient the physical coordinate-axis vectors and how we define the physical vector-product rule.

The notation for the scalar and vector products can be made more compact if instead of $\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}}$, we write $\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3$. We also write $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ as alternate notation for the more general $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$, which need not be orthonormal or even orthogonal. We write the set of physical basis vectors in this notation as $\mathcal{E}$. Thus, if $\mathcal{E}=\{\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3\}$ is orthonormal, we can write in more uniform notation

$$\boldsymbol{v}=v_1\,\hat{\boldsymbol{e}}_1+v_2\,\hat{\boldsymbol{e}}_2+v_3\,\hat{\boldsymbol{e}}_3\,, \tag{27}$$

and the scalar product permits the simple calculation of the components of $\boldsymbol{v}$ as

$$v_k=\hat{\boldsymbol{e}}_k\cdot\boldsymbol{v}\,,\qquad k=1,\,2,\,3\,. \tag{28}$$

On the other hand, if $\mathcal{E}=\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ is arbitrary, we can still write

$$\boldsymbol{v}=v_1\,\boldsymbol{e}_1+v_2\,\boldsymbol{e}_2+v_3\,\boldsymbol{e}_3\,, \tag{29}$$

but the computation of the components is more complicated, namely,

$$v_k=(\boldsymbol{e}_i\times\boldsymbol{e}_j)\cdot\boldsymbol{v}\Big/(\boldsymbol{e}_i\times\boldsymbol{e}_j)\cdot\boldsymbol{e}_k\,.\qquad k=1,\,2,\,3\,, \tag{30}$$

where $i,\ j,\ k$ are cyclic. Equation (30) reduces to (28) if $\mathcal{E}$ is a right-handed orthonormal physical basis.

The orthonormality conditions of (24) and (25) can be written more succinctly as

$$\hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_j = \delta_{ij}\,, \quad i,\, j = 1,\, 2,\, 3\,, \tag{31}$$

where δ_{ij} is the *Kronecker symbol*, defined as

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}\,, \qquad i,\, j = 1,\, 2,\, 3\,. \tag{32}$$

The conditions (26) for right-handedness become likewise

$$\hat{\boldsymbol{e}}_i \cdot (\hat{\boldsymbol{e}}_j \times \hat{\boldsymbol{e}}_k) = \epsilon_{ijk}\,, \qquad i,\, j,\, k = 1,\, 2,\, 3\,, \tag{33}$$

where ϵ_{ijk} is the *Levi-Civita symbol*, defined by

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1\,, \quad \epsilon_{132} = \epsilon_{213} = \epsilon_{321} = -1\,, \tag{34}$$

and all other entries vanish. Thus, ϵ_{ijk} is antisymmetric with respect to the interchange of any two indices,

$$\epsilon_{ijk} = -\epsilon_{jik} = -\epsilon_{ikj} = -\epsilon_{kji}\,, \qquad i,\, j,\, k = 1,\, 2,\, 3\,. \tag{35}$$

We say that ϵ_{ijk} is *totally antisymmetric*. Equation (35) and the value of ϵ_{123} are sufficient to completely specify ϵ_{ijk}.

Equations (26) are equivalent to

$$\hat{\boldsymbol{e}}_i \times \hat{\boldsymbol{e}}_j = \sum_{k=1}^{3} \epsilon_{ijk}\, \hat{\boldsymbol{e}}_k\,, \qquad i,\, j = 1,\, 2,\, 3\,. \tag{36}$$

The Column-Vector Representation

We write the column-vector representation ${}^{\mathcal{E}}\mathbf{v}$ of the *physical* vector $\boldsymbol{v}$ with respect to the *physical* basis $\mathcal{E}$, as

$${}^{\mathcal{E}}\mathbf{v} \equiv \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} {}^{\mathcal{E}}v_1 \\ {}^{\mathcal{E}}v_2 \\ {}^{\mathcal{E}}v_3 \end{bmatrix}\,, \tag{37}$$

where v_i (less ambiguously, ${}^{\mathcal{E}}v_i$), $i = 1,\, 2,\, 3$, is given by (30) (for an orthonormal basis, equivalently by (28)). We can also write more systematically for an arbitrary basis $\mathcal{E}$,

$$\boldsymbol{v} = {}^{\mathcal{E}}v_1\, \boldsymbol{e}_1 + {}^{\mathcal{E}}v_2\, \boldsymbol{e}_2 + {}^{\mathcal{E}}v_3\, \boldsymbol{e}_3\,. \tag{38}$$

The space of column vectors is also a vector space, which we denote by ${}^{\mathcal{E}}\mathcal{V}$.

As a result of (31) and (33), if

$$\boldsymbol{u} = {}^{\mathcal{E}}u_1\, \hat{\boldsymbol{e}}_1 + {}^{\mathcal{E}}u_2\, \hat{\boldsymbol{e}}_2 + {}^{\mathcal{E}}u_3\, \hat{\boldsymbol{e}}_3 \tag{39}$$

and

$$\boldsymbol{v} = {}^{\mathcal{E}}v_1\,\hat{\boldsymbol{e}}_1 + {}^{\mathcal{E}}v_2\,\hat{\boldsymbol{e}}_2 + {}^{\mathcal{E}}v_3\,\hat{\boldsymbol{e}}_3 \tag{40}$$

are physical vectors, and $\mathcal{E}$ is a right-handed orthonormal basis, then

$$\boldsymbol{u} \cdot \boldsymbol{v} = \sum_{i=1}^{3}\sum_{j=1}^{3} \delta_{ij}\,{}^{\mathcal{E}}u_i\,{}^{\mathcal{E}}v_j \tag{41}$$

and

$$\boldsymbol{u} \times \boldsymbol{v} = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \epsilon_{ijk}\,{}^{\mathcal{E}}u_i\,{}^{\mathcal{E}}v_j\,\hat{\boldsymbol{e}}_k\,. \tag{42}$$

Equation (42) can be rewritten in terms of the determinant as

$$\boldsymbol{u} \times \boldsymbol{v} = \begin{vmatrix} \hat{\boldsymbol{e}}_1 & \hat{\boldsymbol{e}}_2 & \hat{\boldsymbol{e}}_3 \\ {}^{\mathcal{E}}u_1 & {}^{\mathcal{E}}u_2 & {}^{\mathcal{E}}u_3 \\ {}^{\mathcal{E}}v_1 & {}^{\mathcal{E}}v_2 & {}^{\mathcal{E}}v_3 \end{vmatrix}\,. \tag{43}$$

We do not necessarily write the presuperscript for the physical basis for a relation that is true for every basis or when the presuperscript would encumber the notation unnecessarily. Note that although the right-handed orthonormal physical basis $\mathcal{E}$ appears explicitly in right-hand sides of (41) and (42), those expressions are independent of the choice of right-handed orthonormal physical basis.

Scalar and Vector Products of Column Vectors

We define the scalar and vector products of the column-vector representations of physical vectors so that, if ${}^{\mathcal{E}}\mathbf{u}$, ${}^{\mathcal{E}}\mathbf{v}$, and ${}^{\mathcal{E}}\mathbf{w}$ are the column-vector representations of the physical vectors $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ with respect to the same arbitrary basis $\mathcal{E}$, then

$${}^{\mathcal{E}}\mathbf{u} \cdot {}^{\mathcal{E}}\mathbf{v} \equiv \boldsymbol{u} \cdot \boldsymbol{v} \tag{44}$$

and

$$({}^{\mathcal{E}}\mathbf{u} \times {}^{\mathcal{E}}\mathbf{v}) \cdot {}^{\mathcal{E}}\mathbf{w} = (\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w}\,. \tag{45}$$

Equation (45) is equivalent to

$${}^{\mathcal{E}}\mathbf{u} \times {}^{\mathcal{E}}\mathbf{v} \equiv {}^{\mathcal{E}}(\mathbf{u} \times \mathbf{v})\,, \tag{46}$$

where ${}^{\mathcal{E}}\mathbf{u} \times {}^{\mathcal{E}}\mathbf{v}$ denotes the vector product of the column-vector representations ${}^{\mathcal{E}}\mathbf{u}$ and ${}^{\mathcal{E}}\mathbf{v}$, while ${}^{\mathcal{E}}(\mathbf{u} \times \mathbf{v})$ denotes the column-vector representation with respect to the basis $\mathcal{E}$ of the physical vector $\boldsymbol{u} \times \boldsymbol{v}$. Obviously, (44) and (45) must be satisfied if the column-vector representation of the scalar and vector products is to make sense. We note trivially also that

$${}^{\mathcal{E}}(\mathbf{u} + \mathbf{v}) = {}^{\mathcal{E}}\mathbf{u} + {}^{\mathcal{E}}\mathbf{v}\,, \tag{47}$$

where the left-hand side denotes the representation with respect to $\mathcal{E}$ of $\boldsymbol{u} + \boldsymbol{v}$, and

$${}^{\mathcal{E}}(c\,\mathbf{u}) = c\,{}^{\mathcal{E}}\mathbf{u}\,, \tag{48}$$

where the left-hand side denotes the representation with respect to $\mathcal{E}$ of $c\boldsymbol{u}$. Equations (44)–(48) show that the operation of representation considered as a map from $\mathcal{V}$, the physical vector space, to ${}^{\mathcal{E}}\mathcal{V}$, the vector space of column-vector representations with respect to $\mathcal{E}$, is a homomorphism [10]. This is equivalent to saying that, if $\mathcal{V}$ is a vector space, then so is ${}^{\mathcal{E}}\mathcal{V}$. The representation map is surjective (onto) and, therefore, also an isomorphism [10].

In more concrete terms, if $\mathcal{E}$ is right-handed orthonormal, then

$$ {}^{\mathcal{E}}\mathbf{u} \cdot {}^{\mathcal{E}}\mathbf{v} = \sum_{i=1}^{3} \sum_{j=1}^{3} \delta_{ij}\, {}^{\mathcal{E}}u_i\, {}^{\mathcal{E}}v_j \tag{49} $$

and

$$ {}^{\mathcal{E}}\mathbf{u} \times {}^{\mathcal{E}}\mathbf{v} \equiv \begin{bmatrix} \left({}^{\mathcal{E}}\mathbf{u} \times {}^{\mathcal{E}}\mathbf{v}\right)_1 \\ \left({}^{\mathcal{E}}\mathbf{u} \times {}^{\mathcal{E}}\mathbf{v}\right)_2 \\ \left({}^{\mathcal{E}}\mathbf{u} \times {}^{\mathcal{E}}\mathbf{v}\right)_3 \end{bmatrix} , \tag{50} $$

with

$$ \left({}^{\mathcal{E}}\mathbf{u} \times {}^{\mathcal{E}}\mathbf{v}\right)_k \equiv \sum_{i=1}^{3} \sum_{j=1}^{3} \epsilon_{ijk}\, {}^{\mathcal{E}}u_i\, {}^{\mathcal{E}}v_j\,, \qquad k = 1,\ 2,\ 3\,. \tag{51} $$

Explicitly,

$$ {}^{\mathcal{E}}\mathbf{u} \cdot {}^{\mathcal{E}}\mathbf{v} = {}^{\mathcal{E}}u_1\, {}^{\mathcal{E}}v_1 + {}^{\mathcal{E}}u_2\, {}^{\mathcal{E}}v_2 + {}^{\mathcal{E}}u_3\, {}^{\mathcal{E}}v_3 \tag{52} $$

and

$$ {}^{\mathcal{E}}\mathbf{u} \times {}^{\mathcal{E}}\mathbf{v} = \begin{bmatrix} {}^{\mathcal{E}}v_2\, {}^{\mathcal{E}}v_3 - {}^{\mathcal{E}}v_3\, {}^{\mathcal{E}}v_2 \\ {}^{\mathcal{E}}v_3\, {}^{\mathcal{E}}v_1 - {}^{\mathcal{E}}v_1\, {}^{\mathcal{E}}v_3 \\ {}^{\mathcal{E}}v_1\, {}^{\mathcal{E}}v_2 - {}^{\mathcal{E}}v_2\, {}^{\mathcal{E}}v_1 \end{bmatrix} , \tag{53} $$

If $\mathbf{u}^{\mathrm{T}}$ denotes the transpose of $\mathbf{u}$, a *row vector*, then the scalar and vector products of column vectors can be written as

$$ {}^{\mathcal{E}}\mathbf{u} \cdot {}^{\mathcal{E}}\mathbf{v} \equiv {}^{\mathcal{E}}\mathbf{u}^{\mathrm{T}}\, {}^{\mathcal{E}}\mathbf{v} \tag{54} $$

and

$$ {}^{\mathcal{E}}\mathbf{u} \times {}^{\mathcal{E}}\mathbf{v} \equiv \left[{}^{\mathcal{E}}\mathbf{u} \times\right] {}^{\mathcal{E}}\mathbf{v}\,, \tag{55} $$

where $[\mathbf{u}\times]$ is the 3×3 antisymmetric matrix given (for any 3×1 array $\mathbf{u}$) by

$$ [\mathbf{u}\times] \equiv \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} \tag{56} $$

or, equivalently,

$$ [\mathbf{u}\times]_{ij} \equiv -\sum_{k=1}^{3} \epsilon_{ijk}\, u_k\,, \qquad i,\ j = 1,\ 2,\ 3\,. \tag{57} $$

The evaluation of (55) leads to the familiar form of the components of the vector product in (53). The row vectors also form a vector space, which we denote by ${}^{\mathcal{E}}\mathcal{V}^{\mathrm{T}}$. Equations

(54) and (55) permit us to write scalar and vector products of any two 3x1 arrays, which may not be the column-vector representations of physical vectors with respect to a common basis. This can be of great practical utility in mission analysis.

The Autorepresentation

We can increase the parallelism between physical vectors and column vectors by considering the *autorepresentation* of a basis, that is, the representation of a basis with respect to itself. Clearly, for every basis $\mathcal{E} = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$, the corresponding autorepresentation

$$^{\mathcal{E}}\mathcal{E} = \{^{\mathcal{E}}\mathbf{e}_1, {}^{\mathcal{E}}\mathbf{e}_2, {}^{\mathcal{E}}\mathbf{e}_3\} \tag{58}$$

must have the values

$$^{\mathcal{E}}\mathbf{e}_1 = \mathbf{1}\,, \qquad {}^{\mathcal{E}}\mathbf{e}_2 = \mathbf{2}\,, \qquad {}^{\mathcal{E}}\mathbf{e}_3 = \mathbf{3}\,, \tag{59}$$

where

$$\mathbf{1} \equiv \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{2} \equiv \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{3} \equiv \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{60}$$

Corresponding to (38), we have

$$\begin{aligned} ^{\mathcal{E}}\mathbf{v} &= {}^{\mathcal{E}}v_1\, {}^{\mathcal{E}}\mathbf{e}_1 + {}^{\mathcal{E}}v_2\, {}^{\mathcal{E}}\mathbf{e}_2 + {}^{\mathcal{E}}v_3\, {}^{\mathcal{E}}\mathbf{e}_3 \\ &= {}^{\mathcal{E}}v_1\, \mathbf{1} + {}^{\mathcal{E}}v_2\, \mathbf{2} + {}^{\mathcal{E}}v_3\, \mathbf{3}\,. \end{aligned} \tag{61}$$

Equations (59)–(61) are true even if the basis $\mathcal{E}$ is not orthonormal. Despite the appearance of (60), the scalar product of any pair of column vectors in (60) need not be the Kronecker symbol, that is, possibly, $\mathbf{1} \cdot \mathbf{1} \neq 1$ and $\mathbf{1} \cdot \mathbf{2} \neq 0$. This unhappy result follows from (44), namely, $^{\mathcal{E}}\mathbf{e}_i \cdot {}^{\mathcal{E}}\mathbf{e}_j = \boldsymbol{e}_i \cdot \boldsymbol{e}_j$, and the fact that, for an arbitrary basis $\mathcal{E}$, we do not have necessarily that $\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}$. Orthonormality is determined not only by the entries in the 3×1 arrays of (60), but also by the rule for the scalar product, and similarly for right-handed orthonormality and the rule for the vector product. The column-vector basis $\{\mathbf{1}, \mathbf{2}, \mathbf{3}\}$, the autorepresentation of $\mathcal{E}$, is orthonormal if and only if the basis $\mathcal{E}$ is orthonormal, and the autorepresentation of $\mathcal{E}$ is right-handed orthonormal if and only if the basis $\mathcal{E}$ is right-handed orthonormal. Hence, no caret appears over $^{\mathcal{E}}\mathbf{e}_k$, $\mathbf{1}$, $\mathbf{2}$, or $\mathbf{3}$ in (58)–(61). We write the caret only if $\mathcal{E}$ is orthonormal.

For a non-orthonormal basis $\mathcal{E}$, we need to replace the Kronecker symbol in the scalar product in (41) and (49) by the more general symmetric symbol $^{\mathcal{E}}g_{ij} \equiv \boldsymbol{e}_i \cdot \boldsymbol{e}_j$, and the Levi-Civita symbol in the vector product in (42) and (51) by the more general totally antisymmetric symbol $^{\mathcal{E}}\Gamma_{ijk} \equiv (\boldsymbol{e}_i \times \boldsymbol{e}_j) \cdot \boldsymbol{e}_k$. Equations (52) and (53) are no longer correct for a basis that is not right-handed orthonormal. In the remainder of this work, $\mathcal{E}$ is always an orthonormal basis (usually also right-handed orthonormal), and we write the caret ($^{\mathcal{E}}\hat{\mathbf{e}}_k$, $\hat{\mathbf{1}}$, $\hat{\mathbf{2}}$, $\hat{\mathbf{3}}$). For an example of an attitude application using a non-orthogonal basis, 11.

The Transformation of Vectors

Physical Rotations

The principal object of this work is to study the transformations of vectors under rotation. Hence, in this section, the purely algebraic must meet the geometrical and tensorial, since rotations are geometrical transformations and the attitude matrix is a tensor. Since the vectors we draw are physical vectors, we consider these first.

Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be two physical vectors of the same magnitude (see Figure 1). Suppose that $\boldsymbol{a}$ is transformed into $\boldsymbol{b}$ by a physical rotation (a *physical* rotation, because it rotates one physical vector into another) through an *angle of rotation* θ about a *physical axis of rotation* $\hat{\boldsymbol{n}}$, which is a physical vector of unit length. We seek an analytical form for this transformation.

Euler's theorem [12, 13] states that every physical rotation leaves one direction unaltered. This direction is the axis of rotation $\hat{\boldsymbol{n}}$, also called the *Euler axis*. Then we can write

$$\boldsymbol{a} = \boldsymbol{a}_{\|} + \boldsymbol{a}_{\perp}\,, \qquad \boldsymbol{b} = \boldsymbol{b}_{\|} + \boldsymbol{b}_{\perp}\,, \tag{62}$$

where

$$\boldsymbol{a}_{\|} = (\hat{\boldsymbol{n}} \cdot \boldsymbol{a})\,\hat{\boldsymbol{n}}\,, \qquad \boldsymbol{b}_{\|} = (\hat{\boldsymbol{n}} \cdot \boldsymbol{b})\,\hat{\boldsymbol{n}}\,, \tag{63}$$

By Euler's theorem, we have

$$\boldsymbol{a}_{\|} = \boldsymbol{b}_{\|}\,, \tag{64}$$

whence,

$$\boldsymbol{a} \cdot \hat{\boldsymbol{n}} = \boldsymbol{b} \cdot \hat{\boldsymbol{n}}\,. \tag{65}$$

From the Grassman identity,

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\,\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\,\boldsymbol{c}\,, \tag{66}$$

it follows that, for every physical vector $\boldsymbol{u}$, that

$$\begin{aligned} \boldsymbol{u} &= (\hat{\boldsymbol{n}} \cdot \boldsymbol{u})\,\hat{\boldsymbol{n}} - \hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \boldsymbol{u}) \\ &\equiv \boldsymbol{u}_{\|} + \boldsymbol{u}_{\perp}\,, \end{aligned} \tag{67}$$

and we can write

$$\boldsymbol{a}_{\perp} = -\hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \boldsymbol{a})\,, \tag{68}$$

$$\boldsymbol{b}_{\perp} = -\hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \boldsymbol{b})\,. \tag{69}$$

The set $\{\hat{\boldsymbol{n}},\, \boldsymbol{a}_{\perp},\, \hat{\boldsymbol{n}} \times \boldsymbol{a}_{\perp}\}$ is an orthogonal, but not necessarily orthonormal, triad of physical vectors, which can act as a basis for the physical vector space. Thus,

$$\boldsymbol{b}_{\perp} = c\,\boldsymbol{a}_{\perp} + s\,\hat{\boldsymbol{n}} \times \boldsymbol{a}_{\perp}\,. \tag{70}$$

Because $\boldsymbol{a}_{\perp}$, $\hat{\boldsymbol{n}} \times \boldsymbol{a}_{\perp}$, and $\boldsymbol{b}_{\perp}$ all have the same magnitude (ρ in Figure 2), it follows that

$$c^2 + s^2 = 1\,, \tag{71}$$

and we can write

$$c = \cos\theta\,, \qquad s = \sin\theta\,. \tag{72}$$

The angle θ is the angle of rotation (see Figure 1). Then

$$\begin{aligned} \boldsymbol{b} &= \boldsymbol{a}_{\parallel} + c\,\boldsymbol{a}_{\perp} + s\,\hat{\boldsymbol{n}} \times \boldsymbol{a}_{\perp} \\ &= (\hat{\boldsymbol{n}} \cdot \boldsymbol{a})\,\hat{\boldsymbol{n}} - c\,\hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \boldsymbol{a}) - s\,\hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \boldsymbol{a})) \\ &= (\hat{\boldsymbol{n}} \cdot \boldsymbol{a})\,\hat{\boldsymbol{n}} - c\,\hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \boldsymbol{a}) + s\,(\hat{\boldsymbol{n}} \times \boldsymbol{a})\,. \end{aligned} \tag{73}$$

Applying the Grassman identity (66) to the second term of (73) results in

$$\begin{aligned} \boldsymbol{b} &= (\cos\theta)\,\boldsymbol{a} + (1-\cos\theta)\,(\hat{\boldsymbol{n}} \cdot \boldsymbol{a})\,\hat{\boldsymbol{n}} + (\sin\theta)\,\hat{\boldsymbol{n}} \times \boldsymbol{a} \\ &= \boldsymbol{a} + (\sin\theta)\,\hat{\boldsymbol{n}} \times \boldsymbol{a} + (1-\cos\theta)\,\hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \boldsymbol{a})\,. \end{aligned} \tag{74}$$

This result is Euler's formula, published in 1775 [14]. The form of (74) is essentially that of Gibbs [15]. The equation is illustrated in Figures 1 and 2.

The Direction-Cosine Matrix

Equation (74) gives the physical rotation through the angle θ about the physical axis of rotation $\hat{\boldsymbol{n}}$ of the physical vector $\boldsymbol{a}$ into the physical vector $\boldsymbol{b}$. Of particular interest for attitude studies is the physical rotation of the physical orthonormal basis $\mathcal{E} = \{\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3\}$, the *prior basis*, into another physical orthonormal basis $\mathcal{E}' = \{\hat{\boldsymbol{e}}'_1, \hat{\boldsymbol{e}}'_2, \hat{\boldsymbol{e}}'_3\}$, the *posterior basis*. The transformation from the physical vector $\boldsymbol{a}$ to the physical vector $\boldsymbol{b}$ is linear. Therefore, the same transformation of prior physical basis vectors must lead to the corresponding posterior physical basis vectors. Following (74), we write

$$\hat{\boldsymbol{e}}'_i = (\cos\theta)\,\hat{\boldsymbol{e}}_i + (1-\cos\theta)\,(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{e}}_i)\,\hat{\boldsymbol{n}} + (\sin\theta)\,\hat{\boldsymbol{n}} \times \hat{\boldsymbol{e}}_i\,, \qquad i = 1,\,2,\,3\,. \tag{75}$$

For any two orthonormal bases $\mathcal{E}$ and $\mathcal{E}'$, not necessarily right-handed, the *direction-cosine matrix* C is defined as

$$C_{ij} \;\equiv\; \hat{\boldsymbol{e}}'_i \cdot \hat{\boldsymbol{e}}_j\,, \qquad i,\, j = 1,\,2,\,3\,, \tag{76}$$

where C_{ij} is the $(i,\ j)$ entry of the 3×3 matrix C. For definiteness we write $\mathsf{C}^{\mathcal{E}'/\mathcal{E}}$ (and $C_{ij}^{\mathcal{E}'/\mathcal{E}}$) displaying the prior basis $\mathcal{E}$ and the posterior basis $\mathcal{E}'$ explicitly, and we can write also $\hat{\boldsymbol{n}}^{\mathcal{E}'/\mathcal{E}}$ and $\theta^{\mathcal{E}'/\mathcal{E}}$. Throughout this work, C (or C_{ij}) without a subscript means $\mathsf{C}^{\mathcal{E}'/\mathcal{E}}$ (or $C_{ij}^{\mathcal{E}'/\mathcal{E}}$).

For the case in which $\mathcal{E}$ and $\mathcal{E}'$ are both right-handed orthonormal bases, C is a rotation matrix. The rotation matrix from the primary reference frame (generally, an inertial frame) to the spacecraft body reference frame is called the *attitude matrix* and is typically denoted by A. More generally, if $\mathcal{E}$ and $\mathcal{E}'$ are any two right-handed orthonormal physical bases, we say that $\mathsf{A}^{\mathcal{E}'/\mathcal{E}}$ is the attitude matrix of $\mathcal{E}'$ relative to $\mathcal{E}$. Our special interest in this work is the attitude matrix. Therefore, we use A henceforth to denote a rotation, and we use C when it is not certain that the two bases are both right-handed orthonormal.

Let A be a rotation matrix. Then, substituting (75) into (76) yields

$$A_{ij} = (\cos\theta)\,\delta_{ij} + (1-\cos\theta)\,(\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{e}}_i)(\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{e}}_j) \\ + (\sin\theta)\,(\hat{\boldsymbol{n}}\times\hat{\boldsymbol{e}}_i)\cdot\hat{\boldsymbol{e}}_j\,, \quad i,\,j=1,\,2,\,3\,. \tag{77}$$

Noting

$$(\hat{\boldsymbol{n}}\times\hat{\boldsymbol{e}}_i)\cdot\hat{\boldsymbol{e}}_j = (\hat{\mathbf{n}}\times\hat{\mathbf{e}}_i)\cdot\hat{\mathbf{e}}_j = \hat{\mathbf{e}}_j^{\mathrm{T}}\,[\hat{\mathbf{n}}\times]\,\hat{\mathbf{e}}_i \\ = [\hat{\mathbf{n}}\times]_{ji} = -[\hat{\mathbf{n}}\times]_{ij}\,, \qquad i,\,j=1,\,2,\,3\,, \tag{78}$$

leads to

$$A_{ij} = (\cos\theta)\,\delta_{ij} + (1-\cos\theta)\,n_i\,n_j - (\sin\theta)\,[\hat{\mathbf{n}}\times]_{ij} \\ = \delta_{ij} - (\sin\theta)\,[\hat{\mathbf{n}}\times]_{ij} + (1-\cos\theta)\left([\hat{\mathbf{n}}\times]^2\right)_{ij}\,, \qquad i,\,j=1,\,2,\,3\,. \tag{79}$$

In matrix notation,

$$\mathsf{A} = (\cos\theta)\,\mathsf{I}_{3\times3} + (1-\cos\theta)\,\hat{\mathbf{n}}\hat{\mathbf{n}}^{\mathrm{T}} - (\sin\theta)\,[\hat{\mathbf{n}}\times] \\ = \mathsf{I}_{3\times3} - (\sin\theta)\,[\hat{\mathbf{n}}\times] + (1-\cos\theta)\,[\hat{\mathbf{n}}\times]^2\,, \tag{80}$$

where $\mathsf{I}_{3\times3}$ is the 3×3 identity matrix given by

$$\mathsf{I}_{3\times3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\,, \tag{81}$$

and the antisymmetric matrix $[\hat{\mathbf{n}}\times]$ is defined by (56). In terms of individual entries, (80) can be written as

$$\mathsf{A} = \begin{bmatrix} c + n_1^2\,(1-c) & n_1\,n_2\,(1-c) + n_3\,s & n_1\,n_3\,(1-c) - n_2\,s \\ n_2\,n_1\,(1-c) - n_3\,s & c + n_2^2\,(1-c) & n_2\,n_3\,(1-c) + n_1\,s \\ n_3\,n_1\,(1-c) + n_2\,s & n_3\,n_2\,(1-c) - n_1\,s & c + n_3^2\,(1-c) \end{bmatrix}\,, \tag{82}$$

where $c = \cos\theta$ and $s = \sin\theta$.

It follows from Euler's formula (80) for the attitude matrix that

$$\mathsf{A}(-\hat{\mathbf{n}},\,-\theta) = \mathsf{A}(\hat{\mathbf{n}},\,\theta) \tag{83}$$

and

$$\mathsf{A}(\hat{\mathbf{n}},\,-\theta) = \mathsf{A}(-\hat{\mathbf{n}},\,\theta) = \mathsf{A}^{\mathrm{T}}(\hat{\mathbf{n}},\,\theta)\,. \tag{84}$$

From Euler's theorem, the column-vector representation of the axis of rotation may be with respect to either the prior or the posterior basis, because, from (80),

$$^{\mathcal{E}}\hat{\mathbf{n}} = {}^{\mathcal{E}'}\hat{\mathbf{n}}\,. \tag{85}$$

From (76), we have for the general direction-cosine matrix,

$$\hat{\boldsymbol{e}}'_i = \sum_{j=1}^{3} C_{ij}^{\mathcal{E}'/\mathcal{E}}\, \hat{\boldsymbol{e}}_j\,, \qquad i = 1,\, 2,\, 3\,, \tag{86}$$

$$\hat{\boldsymbol{e}}_i = \sum_{j=1}^{3} C_{ij}^{\mathcal{E}/\mathcal{E}'}\, \hat{\boldsymbol{e}}'_j\,, \qquad i = 1,\, 2,\, 3\,. \tag{87}$$

Equations (86) and (87) describe the transformation of the basis $\mathcal{E}$ into the basis $\mathcal{E}'$ and the inverse transformation. It follows from substituting (87) into (86) and *vice versa* that

$$\sum_{k=1}^{3} C_{ik}^{\mathcal{E}'/\mathcal{E}}\, C_{kj}^{\mathcal{E}/\mathcal{E}'} = \sum_{k=1}^{3} C_{ik}^{\mathcal{E}/\mathcal{E}'}\, C_{kj}^{\mathcal{E}'/\mathcal{E}} = \delta_{ij} \qquad i,\, j = 1,\, 2,\, 3\,. \tag{88}$$

From (76),

$$C_{kj}^{\mathcal{E}/\mathcal{E}'} = C_{jk}^{\mathcal{E}'/\mathcal{E}} = \left((C^{\mathcal{E}'/\mathcal{E}})^{\mathrm{T}}\right)_{kj}\,, \qquad j, k = 1,\, 2,\, 3\,, \tag{89}$$

and similarly for $C_{ik}^{\mathcal{E}/\mathcal{E}'}$. Therefore, we can write in general for the direction-cosine matrix

$$\mathsf{C}\mathsf{C}^{\mathrm{T}} = \mathsf{C}^{\mathrm{T}}\mathsf{C} = \mathsf{I}_{3\times 3}\,. \tag{90}$$

It follows that

$$\det(\mathsf{C}\mathsf{C}^{\mathrm{T}}) = (\det \mathsf{C})(\det \mathsf{C}^{\mathrm{T}}) = (\det \mathsf{C})^2 = 1\,, \tag{91}$$

whence,

$$\det \mathsf{C} = \pm 1\,. \tag{92}$$

Equation (90) is the definition of an *orthogonal* matrix. An orthogonal matrix with determinant $+1$ is called *proper orthogonal*, with determinant -1, *improper orthogonal*. We call a transformation, either of physical vectors or of column vectors, that has an orthogonal transformation matrix an *orthogonal transformation.* Likewise, we may speak of *proper orthogonal transformations* (rotations) and *improper orthogonal transformations*.

For a rotation, the determinant of the direction-cosine matrix must be a continuous function of the angle of rotation. Since, from (80), this determinant is necessarily $+1$ for $\theta = 0$, it follows for the case of a rotation that only the positive sign is possible for all values of the angle of rotation. Thus,

$$\det \mathsf{A} = +1\,. \tag{93}$$

It is easy to show that, given (86), for an orthogonal transformation of an orthonormal basis $\mathcal{E}$ into an orthonormal basis $\mathcal{E}'$, $\hat{\boldsymbol{e}}'_i \cdot \hat{\boldsymbol{e}}'_j = \hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_j$ and $\hat{\boldsymbol{e}}'_i \cdot (\hat{\boldsymbol{e}}'_j \times \hat{\boldsymbol{e}}'_k) = (\det\, \mathsf{C}^{\mathcal{E}'/\mathcal{E}})\, \hat{\boldsymbol{e}}_i \cdot (\hat{\boldsymbol{e}}_j \times \hat{\boldsymbol{e}}_k)$. Thus, an orthogonal matrix carries an orthonormal basis into an orthonormal basis and a proper orthogonal matrix carries a right-handed orthonormal basis into a right-handed orthonormal basis.

Transformation of Vector Representations

Let ${}^{\mathcal{E}}\mathbf{v}$ be the column-vector representation of a physical vector $\boldsymbol{v}$ with respect to the prior orthonormal basis $\mathcal{E}$, and let ${}^{\mathcal{E}'}\mathbf{v}$ be the column-vector representation of this same physical vector with respect to the posterior orthonormal basis $\mathcal{E}'$. Then

$$ {}^{\mathcal{E}'}v_i = \hat{\boldsymbol{e}}'_i \cdot \boldsymbol{v} = \left(\sum_{j=1}^{3} C_{ij}^{\mathcal{E}'/\mathcal{E}} \, \hat{\boldsymbol{e}}_j \right) \cdot \boldsymbol{v} = \sum_{j=1}^{3} C_{ij}^{\mathcal{E}'/\mathcal{E}} \; {}^{\mathcal{E}}v_j \,, \qquad i = 1,\, 2,\, 3\,, \tag{94} $$

or

$$ {}^{\mathcal{E}'}\mathbf{v} = \mathsf{C}^{\mathcal{E}'/\mathcal{E}} \; {}^{\mathcal{E}}\mathbf{v} \,. \tag{95} $$

Applied to the representation of a prior physical basis vector with respect to the prior physical basis, (95) yields

$$ {}^{\mathcal{E}'}\hat{\mathbf{e}}_i = \mathsf{C}^{\mathcal{E}'/\mathcal{E}} \; {}^{\mathcal{E}}\hat{\mathbf{e}}_i \,, \qquad i = 1,\, 2,\, 3\,, \tag{96} $$

which looks similar to

$$ {}^{\mathcal{E}}\hat{\mathbf{e}}'_i = \sum_{j=1}^{3} C_{ij}^{\mathcal{E}'/\mathcal{E}} \; {}^{\mathcal{E}}\hat{\mathbf{e}}_j \,, \qquad i = 1,\, 2,\, 3\,. \tag{97} $$

The characters of the right-hand sides of (96) and (97) are very different. One displays matrix multiplication, and the other displays multiplication of a column vector by a scalar.

Writing the column vector together with the basis of the representation in order to avoid ambiguity is usually a good idea. Examine the following three equations, which hold for a rotation through an angle θ about the physical axis $\hat{\boldsymbol{e}}_3$ (see Figure 3).

$$ {}^{\mathcal{E}}\hat{\mathbf{e}}'_1 = (\cos\theta) \; {}^{\mathcal{E}}\hat{\mathbf{e}}_1 + (\sin\theta) \; {}^{\mathcal{E}}\hat{\mathbf{e}}_2 \,, \tag{98} $$

$$ {}^{\mathcal{E}'}\hat{\mathbf{e}}_1 = (\cos\theta) \; {}^{\mathcal{E}}\hat{\mathbf{e}}_1 - (\sin\theta) \; {}^{\mathcal{E}}\hat{\mathbf{e}}_2 \,, \tag{99} $$

$$ {}^{\mathcal{E}'}\hat{\mathbf{e}}'_1 = {}^{\mathcal{E}}\hat{\mathbf{e}}_1 \,. \tag{100} $$

In different contexts, any one of these left-hand sides might be called $\hat{\mathbf{e}}'_1$.

For $\mathcal{E}$ and $\mathcal{E}'$ orthonormal bases, the identity matrix can be written as

$$ \mathsf{I}_{3\times 3} = \sum_{k=1}^{3} {}^{\mathcal{E}}\hat{\mathbf{e}}_k \, {}^{\mathcal{E}}\hat{\mathbf{e}}_k^{\mathrm{T}} = \sum_{k=1}^{3} {}^{\mathcal{E}'}\hat{\mathbf{e}}'_k \, {}^{\mathcal{E}'}\hat{\mathbf{e}}_k'^{\mathrm{T}} \,, \tag{101} $$

and applying (96)–(97) yields

$$ \mathsf{I}_{3\times 3} = \sum_{k=1}^{3} {}^{\mathcal{E}'}\hat{\mathbf{e}}_k \; {}^{\mathcal{E}'}\hat{\mathbf{e}}_k^{\mathrm{T}} = \sum_{k=1}^{3} {}^{\mathcal{E}}\hat{\mathbf{e}}'_k \; {}^{\mathcal{E}}\hat{\mathbf{e}}_k'^{\mathrm{T}} \,. \tag{102} $$

It follows that

$$
\begin{aligned}
{}^{\mathcal{E}'}\hat{\mathbf{e}}_i &= \sum_{j=1}^{3} {}^{\mathcal{E}'}\hat{\mathbf{e}}'_j \, {}^{\mathcal{E}'}\hat{\mathbf{e}}'^{\mathrm{T}}_j \, {}^{\mathcal{E}}\hat{\mathbf{e}}_i \\
&= \sum_{j=1}^{3} (\hat{\boldsymbol{e}}'_j \cdot \hat{\boldsymbol{e}}_i) \, {}^{\mathcal{E}'}\hat{\mathbf{e}}'_j = \sum_{j=1}^{3} \mathsf{C}^{\mathcal{E}'/\mathcal{E}}_{ji} \, {}^{\mathcal{E}'}\hat{\mathbf{e}}'_j \\
&= \sum_{j=1}^{3} \mathsf{C}^{\mathcal{E}'/\mathcal{E}}_{ji} \, {}^{\mathcal{E}}\hat{\mathbf{e}}_j = \sum_{j=1}^{3} \left((\mathsf{C}^{\mathcal{E}'/\mathcal{E}})^{\mathrm{T}} \right)_{ij} {}^{\mathcal{E}}\hat{\mathbf{e}}_j \, .
\end{aligned} \tag{103}
$$

In (101)–(103)we use the fact that ${}^{\mathcal{E}}\hat{\mathbf{e}}_k$ and ${}^{\mathcal{E}'}\hat{\mathbf{e}}'_k$, $k = 1, 2, 3$, are elements of the autorepresentation of $\mathcal{E}$ and $\mathcal{E}'$, that is, the numerical vectors $\hat{\mathbf{1}}$, $\hat{\mathbf{2}}$, and $\hat{\mathbf{3}}$. Hence,

$$
{}^{\mathcal{E}}\hat{\mathbf{e}}_k = {}^{\mathcal{E}'}\hat{\mathbf{e}}'_k \, , \qquad k = 1, 2, 3 \, . \tag{104}
$$

The subscript of $\hat{\mathbf{e}}_j$ in (103) is not a matrix index, and the multiplication operation is not matrix multiplication. Equation (103) together with (87) clarifies the different signs in (98) and (99).

Alternatives Forms of the Direction-Cosine Matrix

Let $\mathcal{A} = \{\hat{\boldsymbol{a}}_1, \hat{\boldsymbol{a}}_2, \hat{\boldsymbol{a}}_3\}$, $\mathcal{B} = \{\hat{\boldsymbol{b}}_1, \hat{\boldsymbol{b}}_2, \hat{\boldsymbol{b}}_3\}$, and $\mathcal{C} = \{\hat{\boldsymbol{c}}_1, \hat{\boldsymbol{c}}_2, \hat{\boldsymbol{c}}_3\}$ be three orthonormal bases. Then we can write the direction-cosine matrix of the transformation from $\mathcal{A}$ to $\mathcal{B}$ as

$$
C^{\mathcal{B}/\mathcal{A}}_{ij} = \hat{\boldsymbol{b}}_i \cdot \hat{\boldsymbol{a}}_j = {}^{\mathcal{A}}\hat{\mathbf{b}}^{\mathrm{T}}_i \, {}^{\mathcal{A}}\hat{\mathbf{a}}_j = {}^{\mathcal{B}}\hat{\mathbf{b}}^{\mathrm{T}}_i \, {}^{\mathcal{B}}\hat{\mathbf{a}}_j \, , \qquad i, j = 1, 2, 3 \, . \tag{105}
$$

Hence, noting (59),

$$
\hat{\boldsymbol{b}}_i \cdot \hat{\boldsymbol{a}}_j = \left({}^{\mathcal{B}}\hat{\mathbf{a}}_j \right)_i = \left({}^{\mathcal{A}}\hat{\mathbf{b}}_i \right)_j \, , \qquad i, j = 1, 2, 3 \, , \tag{106}
$$

where, for example, the expression between the two equal signs of (106) denotes the i-th component of the representation with respect to $\mathcal{B}$ of $\hat{\boldsymbol{a}}_j$. Thus,

$$
\begin{aligned}
\mathsf{C}^{\mathcal{B}/\mathcal{A}} &= \sum_{k=1}^{3} {}^{\mathcal{A}}\hat{\mathbf{a}}_k \, {}^{\mathcal{A}}\hat{\mathbf{b}}^{\mathrm{T}}_k = [\, {}^{\mathcal{A}}\hat{\mathbf{b}}_1 \quad {}^{\mathcal{A}}\hat{\mathbf{b}}_2 \quad {}^{\mathcal{A}}\hat{\mathbf{b}}_3 \,]^{\mathrm{T}} \\
&= \sum_{k=1}^{3} {}^{\mathcal{B}}\hat{\mathbf{a}}_k \, {}^{\mathcal{B}}\hat{\mathbf{b}}^{\mathrm{T}}_k = [\, {}^{\mathcal{B}}\hat{\mathbf{a}}_1 \quad {}^{\mathcal{B}}\hat{\mathbf{a}}_2 \quad {}^{\mathcal{B}}\hat{\mathbf{a}}_3 \,] \, .
\end{aligned} \tag{107}
$$

Because $\mathcal{C}$ is an orthonormal basis, it follows that

$$
\sum_{k=1}^{3} {}^{\mathcal{E}}\hat{\mathbf{c}}_k \, {}^{\mathcal{E}}\hat{\mathbf{c}}^{\mathrm{T}}_k = \mathsf{I}_{3\times 3} \tag{108}
$$

for every representation of the basis column vectors of C with respect to a common basis $\mathcal{E}$. Then

$$
\begin{aligned}
C_{ij}^{\mathcal{B}/\mathcal{A}} &= {}^{C}\hat{\mathbf{b}}_i^{\mathrm{T}}\, {}^{C}\hat{\mathbf{a}}_j = {}^{C}\hat{\mathbf{b}}_i^{\mathrm{T}} \left(\sum_{k=1}^{3} {}^{C}\hat{\mathbf{c}}_k \, {}^{C}\hat{\mathbf{c}}_k^{\mathrm{T}} \right) {}^{C}\hat{\mathbf{a}}_j \\
&= \sum_{k=1}^{3} {}^{C}\hat{\mathbf{b}}_i^{\mathrm{T}} \, {}^{C}\hat{\mathbf{c}}_k \, {}^{C}\hat{\mathbf{c}}_k^{\mathrm{T}} \, {}^{C}\hat{\mathbf{a}}_j \\
&= \sum_{k=1}^{3} (\hat{\mathbf{b}}_i \cdot \hat{\boldsymbol{c}}_k)(\hat{\boldsymbol{c}}_k \cdot \hat{\mathbf{a}}_j) \\
&= \sum_{k=1}^{3} \left({}^{\mathcal{B}}\hat{\mathbf{c}}_k\right)_i \left({}^{\mathcal{A}}\hat{\mathbf{c}}_k\right)_j^{\mathrm{T}}, \qquad i,\ j = 1,\ 2,\ 3
\end{aligned}
\tag{109}
$$

or

$$
\mathsf{C}^{\mathcal{B}/\mathcal{A}} = \sum_{k=1}^{3} {}^{\mathcal{B}}\hat{\mathbf{c}}_k \, {}^{\mathcal{A}}\hat{\mathbf{c}}_k^{\mathrm{T}} . \tag{110}
$$

Equation (110) bears a close resemblance to the result for the TRIAD algorithm [5, 16] in attitude estimation.

The Rotation Vector and Infinitesimal Rotations

The column vector

$$
\boldsymbol{\theta} \equiv \theta \, \hat{\mathbf{n}} \tag{111}
$$

is called the *rotation vector*. Like the axis and angle of rotation, and the attitude matrix, the rotation vector is also a representation of the attitude. (An *attitude representation* is a set of parameters from which the attitude matrix can be calculated.) The most important use of the rotation vector is as an infinitesimal quantity $\Delta\boldsymbol{\theta}$, that is, when $|\Delta\boldsymbol{\theta}|^2$ is so small as to be insignificant compared to $|\Delta\boldsymbol{\theta}|$. In that case, we can write the attitude matrix as

$$
\begin{aligned}
\delta\mathsf{A} &= \mathsf{I}_{3\times3} - [\, \Delta\boldsymbol{\theta}\times \,] + O(|\Delta\boldsymbol{\theta}|^2) \\
&\approx \mathsf{I}_{3\times3} - [\, \Delta\boldsymbol{\theta}\times \,] ,
\end{aligned}
\tag{112}
$$

where $\delta\mathsf{A}$ indicates an *infinitesimal rotation*, that is, a rotation through an infinitesimal angle $\Delta\theta$. $\Delta\boldsymbol{\theta}$ finds the most useful application as an attitude correction or as an attitude error. For $\boldsymbol{\theta} \neq \mathbf{0}$, we can write

$$
\begin{aligned}
\mathsf{A}(\boldsymbol{\theta}) &= (\cos|\boldsymbol{\theta}|)\ \mathsf{I}_{3\times3} - \frac{\sin|\boldsymbol{\theta}|}{|\boldsymbol{\theta}|} [\, \boldsymbol{\theta}\times \,] + \frac{(1-\cos|\boldsymbol{\theta})|)}{|\boldsymbol{\theta}|^2} \boldsymbol{\theta}\,\boldsymbol{\theta}^{\mathrm{T}} \\
&= \mathsf{I}_{3\times3} - \frac{\sin|\boldsymbol{\theta}|}{|\boldsymbol{\theta}|} [\, \boldsymbol{\theta}\times \,] + \frac{(1-\cos|\boldsymbol{\theta})|)}{|\boldsymbol{\theta}|^2} [\, \boldsymbol{\theta}\times \,]^2 .
\end{aligned}
\tag{113}
$$

In more detailed notation, we can write $\theta^{\mathcal{E}'/\mathcal{E}}$. One could construct a *physical rotation vector* as $\theta\hat{\boldsymbol{n}}$, but it is not a very useful quantity, except, perhaps, for writing expressions for the attitude dyadic (see below) that are analogous to those for the attitude matrix. Since there is no practical need for a physical rotation vector, we speak generally of the rotation vector and not of the rotation column vector.

Dual Vectors and Dyadics

Dual Vectors

Let us define the *physical dual vector* $\boldsymbol{u}^\dagger$, which is dual to the physical vector $\boldsymbol{u}$ as the linear operator satisfying

$$\boldsymbol{u}^\dagger \boldsymbol{v} = \boldsymbol{u} \cdot \boldsymbol{v} \tag{114}$$

for every physical vector $\boldsymbol{v} \in \mathcal{V}$. The space $\mathcal{V}^\dagger$ of physical dual vectors is also a vector space, the *dual space*. The dual vector $({}^{\mathcal{E}}\mathbf{u})^\dagger$ of the column vector ${}^{\mathcal{E}}\mathbf{u}$ is the row vector ${}^{\mathcal{E}}\mathbf{u}^{\mathrm{T}}$, and we have

$$\boldsymbol{u}^\dagger \boldsymbol{v} = \boldsymbol{u} \cdot \boldsymbol{v} = {}^{\mathcal{E}}\mathbf{u}^{\mathrm{T}}\, {}^{\mathcal{E}}\mathbf{v}\,. \tag{115}$$

We can define a scalar product in the dual space by $\boldsymbol{u}^\dagger \cdot \boldsymbol{v}^\dagger \equiv \boldsymbol{u} \cdot \boldsymbol{v}$, from which, $(\boldsymbol{u}^\dagger)^\dagger \boldsymbol{v}^\dagger = \boldsymbol{u}^\dagger \cdot \boldsymbol{v}^\dagger = \boldsymbol{u} \cdot \boldsymbol{v}$, and therefore, $(\boldsymbol{u}^\dagger)^\dagger = \boldsymbol{u}$. Thus, the dual vector of the dual vector is the vector itself [9].

Dyadics

The dual vector provides a convenient notation for describing linear operators. For $\mathcal{E}$ an orthonormal basis, we can write an arbitrary 3×3 matrix as

$${}^{\mathcal{E}}\mathsf{F} = \sum_{i=1}^{3} \sum_{j=1}^{3} {}^{\mathcal{E}}F_{ij}\, {}^{\mathcal{E}}\hat{\mathbf{e}}_i\, {}^{\mathcal{E}}\hat{\mathbf{e}}_j^{\mathrm{T}} = \begin{bmatrix} {}^{\mathcal{E}}F_{11} & {}^{\mathcal{E}}F_{12} & {}^{\mathcal{E}}F_{13} \\ {}^{\mathcal{E}}F_{21} & {}^{\mathcal{E}}F_{22} & {}^{\mathcal{E}}F_{23} \\ {}^{\mathcal{E}}F_{31} & {}^{\mathcal{E}}F_{32} & {}^{\mathcal{E}}F_{33} \end{bmatrix}\,. \tag{116}$$

Trivially,

$${}^{\mathcal{E}}F_{ij} = {}^{\mathcal{E}}\hat{\mathbf{e}}_i^{\mathrm{T}}\, {}^{\mathcal{E}}\mathsf{F}\, {}^{\mathcal{E}}\hat{\mathbf{e}}_j\,, \qquad i,\, j = 1,\, 2,\, 3\,. \tag{117}$$

Note that ${}^{\mathcal{E}}\hat{\mathbf{e}}_k$, $k = 1, 2, 3$, in (116) and (117) are elements of the autorepresentation of $\mathcal{E}$.

The matrix ${}^{\mathcal{E}}\mathsf{F}$ transforms under a change of basis according to

$${}^{\mathcal{E}'}\mathsf{F} = \mathsf{C}^{\mathcal{E}'/\mathcal{E}}\, {}^{\mathcal{E}}\mathsf{F} \left(\mathsf{C}^{\mathcal{E}'/\mathcal{E}}\right)^{\mathrm{T}} \tag{118}$$

or, equivalently, according to

$${}^{\mathcal{E}'}\mathsf{F}_{ij} = \sum_{k=1}^{3} \sum_{m=1}^{3} \mathsf{C}_{ik}^{\mathcal{E}'/\mathcal{E}}\, {}^{\mathcal{E}}\mathsf{F}_{km} \left(\left(\mathsf{C}^{\mathcal{E}'/\mathcal{E}}\right)^{\mathrm{T}}\right)_{mj}\,. \tag{119}$$

To see that this transformation is consistent with (116), we insert (116) into (118) to obtain

$$
\begin{aligned}
{}^{\mathcal{E}'}\mathsf{F} &= \sum_{i=1}^{3}\sum_{j=1}^{3} {}^{\mathcal{E}}\mathsf{F}_{ij} \left(\mathsf{C}^{\mathcal{E}'/\mathcal{E}}\, {}^{\mathcal{E}}\hat{\mathbf{e}}_i\right) \left(\mathsf{C}^{\mathcal{E}'/\mathcal{E}}\, {}^{\mathcal{E}}\hat{\mathbf{e}}_j\right)^{\mathrm{T}} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} {}^{\mathcal{E}}\mathsf{F}_{ij}\, {}^{\mathcal{E}'}\hat{\mathbf{e}}_i\, {}^{\mathcal{E}'}\hat{\mathbf{e}}_j^{\mathrm{T}} .
\end{aligned} \tag{120}
$$

Applying (103) to (120)

$$
\begin{aligned}
{}^{\mathcal{E}'}\mathsf{F} &= \sum_{i=1}^{3}\sum_{j=1}^{3} {}^{\mathcal{E}}\mathsf{F}_{ij} \left(\sum_{k=1}^{3} C_{ki}^{\mathcal{E}'/\mathcal{E}}\, {}^{\mathcal{E}'}\hat{\mathbf{e}}'_k\right) \left(\sum_{m=1}^{3} C_{mj}^{\mathcal{E}'/\mathcal{E}}\, {}^{\mathcal{E}'}\hat{\mathbf{e}}'_m\right)^{\mathrm{T}} \\
&= \sum_{k=1}^{3}\sum_{m=1}^{3}\sum_{i=1}^{3}\sum_{j=1}^{3} C_{ki}^{\mathcal{E}'/\mathcal{E}}\, {}^{\mathcal{E}}\mathsf{F}_{ij} \left(\left(C^{\mathcal{E}'/\mathcal{E}}\right)^{\mathrm{T}}\right)_{jm} {}^{\mathcal{E}'}\hat{\mathbf{e}}'_k\, {}^{\mathcal{E}'}\hat{\mathbf{e}}_m'^{\mathrm{T}} \\
&= \sum_{k=1}^{3}\sum_{m=1}^{3} {}^{\mathcal{E}'}\mathsf{F}_{km}\, {}^{\mathcal{E}'}\hat{\mathbf{e}}'_k\, {}^{\mathcal{E}'}\hat{\mathbf{e}}_m'^{\mathrm{T}} ,
\end{aligned} \tag{121}
$$

which is identical in form to (116).

We define the corresponding physical-vector-valued *linear operator* on the physical vector space $\mathcal{V}$ as the bilinear form

$$
\mathcal{F} \equiv \sum_{i=1}^{3}\sum_{j=1}^{3} {}^{\mathcal{E}}F_{ij}\, \hat{\boldsymbol{e}}_i\, \hat{\boldsymbol{e}}_j^{\dagger} , \tag{122}
$$

and therefore,

$$
{}^{\mathcal{E}}F_{ij} = \hat{\boldsymbol{e}}_i^{\dagger}\, \mathcal{F}\, \hat{\boldsymbol{e}}_j , \qquad i,\ j = 1,\ 2,\ 3 . \tag{123}
$$

The bilinear form in (122) is called a *dyadic*. The dual vector and the dyadic make possible a parallelism between the vector space of physical vectors and physical operators and the vector space of column vectors and matrices, namely,

$$
\boldsymbol{v},\ \boldsymbol{v}^{\dagger},\ \mathcal{F} \leftrightarrow\ {}^{\mathcal{E}}\mathbf{v},\ {}^{\mathcal{E}}\mathbf{v}^{\mathrm{T}},\ {}^{\mathcal{E}}\mathsf{F} . \tag{124}
$$

Note that $\mathcal{F}$ is frame independent. ${}^{\mathcal{E}}\mathsf{F}$ depends on the choice of frame (namely, $\mathcal{E}$). Note also from (117) and (123) that

$$
\hat{\boldsymbol{e}}_i^{\dagger}\, \mathcal{F}\, \hat{\boldsymbol{e}}_j = {}^{\mathcal{E}}\hat{\mathbf{e}}_i^{\mathrm{T}}\, {}^{\mathcal{E}}\mathsf{F}\, {}^{\mathcal{E}}\hat{\mathbf{e}}_j , \qquad i,\ j = 1,\ 2,\ 3 \tag{125}
$$

for every orthonormal basis $\mathcal{E}$.

To the transpose matrix ${}^{\mathcal{E}}\mathsf{F}^{\mathrm{T}}$ corresponds the *conjugate dyadic* $\mathcal{F}^{\dagger}$ given by

$$
\mathcal{F}^{\dagger} = \sum_{i=1}^{3}\sum_{j=1}^{3} \left({}^{\mathcal{E}}F^{\mathrm{T}}\right)_{ij} \hat{\boldsymbol{e}}_i\, \hat{\boldsymbol{e}}_j^{\dagger} = \sum_{i=1}^{3}\sum_{j=1}^{3} {}^{\mathcal{E}}F_{ji}\, \hat{\boldsymbol{e}}_i\, \hat{\boldsymbol{e}}_j^{\dagger} = \sum_{i=1}^{3}\sum_{j=1}^{3} {}^{\mathcal{E}}F_{ij}\, \hat{\boldsymbol{e}}_j\, \hat{\boldsymbol{e}}_i^{\dagger} . \tag{126}
$$

Similarly to the transposition of matrices, we have

$$\left(\mathcal{F}\mathcal{G}\right)^{\dagger} = \mathcal{G}^{\dagger}\mathcal{F}^{\dagger} . \tag{127}$$

From (56) or (57), we can write the matrix $[{}^{\mathcal{E}}\mathbf{u}\times]$ as

$$[\,{}^{\mathcal{E}}\mathbf{u}\times\,] = \sum_{i=1}^{3}\sum_{j=1}^{3}[{}^{\mathcal{E}}\mathbf{u}\times]_{ij}\,{}^{\mathcal{E}}\hat{\mathbf{e}}_i\,{}^{\mathcal{E}}\hat{\mathbf{e}}_j^{\mathrm{T}} = -\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\epsilon_{ijk}\,{}^{\mathcal{E}}u_i\,{}^{\mathcal{E}}\hat{\mathbf{e}}_j\,{}^{\mathcal{E}}\hat{\mathbf{e}}_k^{\mathrm{T}} , \tag{128}$$

and we can write the corresponding dyadic as

$$\{\,\boldsymbol{u}\times\,\} = \sum_{i=1}^{3}\sum_{j=1}^{3}[\,{}^{\mathcal{E}}\mathbf{u}\times]_{ij}\,\hat{\boldsymbol{e}}_i\,\hat{\boldsymbol{e}}_j^{\dagger} = -\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\epsilon_{ijk}\,{}^{\mathcal{E}}u_i\,\hat{\boldsymbol{e}}_j\,\hat{\boldsymbol{e}}_k^{\dagger} , \tag{129}$$

hence,

$$\{\boldsymbol{u}\times\}\,\boldsymbol{v} = \boldsymbol{u}\times\boldsymbol{v} . \tag{130}$$

Despite the appearance of a particular basis in the central and right-hand sides of (129), the dyadic $\{\boldsymbol{u}\times\}$ is independent of any representation. The identity dyadic has the form analogous to (108)

$$\mathcal{I} = \sum_{k=1}^{3}\hat{\boldsymbol{e}}_k\,\hat{\boldsymbol{e}}_k^{\dagger} , \tag{131}$$

which satisfies

$$\mathcal{I}\boldsymbol{v} = \boldsymbol{v} \tag{132}$$

for every $\boldsymbol{v} \in \mathcal{V}$. Reference [1] defines dyadics in the classical manner, without the introduction of dual vectors.

The Attitude Dyadic

The attitude matrix satisfies

$$\mathsf{A}^{\mathcal{E}'/\mathcal{E}}\ {}^{\mathcal{E}}\hat{\mathbf{e}}_k = {}^{\mathcal{E}'}\hat{\mathbf{e}}_k , \qquad k = 1,\,2,\,3 . \tag{133}$$

A presuperscript on $\mathsf{A}^{\mathcal{E}'/\mathcal{E}}$ would be superfluous, because the prior and posterior bases determine the value of the attitude matrix unambiguously.

From (133), it follows that

$$\mathsf{A}^{\mathcal{E}'/\mathcal{E}} = \sum_{k=1}^{3}{}^{\mathcal{E}'}\hat{\mathbf{e}}_k\,{}^{\mathcal{E}}\hat{\mathbf{e}}_k^{\mathrm{T}} . \tag{134}$$

We cannot construct the attitude dyadic by means of trivial substitution of physical basis vectors for their autorepresentations in (134), the procedure in passing from (116) to

(122), because the representations in (134) are not all with respect to the same basis. We begin, therefore, by writing

$$
\begin{aligned}
{}^{\mathcal{E}}\hat{\mathbf{e}}_i^{\mathrm{T}}\ \mathsf{A}^{\mathcal{E}'/\mathcal{E}}\ {}^{\mathcal{E}}\hat{\mathbf{e}}_j &= \sum_{k=1}^{3} \left({}^{\mathcal{E}}\hat{\mathbf{e}}_i^{\mathrm{T}}\, {}^{\mathcal{E}'}\hat{\mathbf{e}}_k\right) \left({}^{\mathcal{E}}\hat{\mathbf{e}}_k^{\mathrm{T}}\, {}^{\mathcal{E}}\hat{\mathbf{e}}_j\right) \\
&= {}^{\mathcal{E}}\hat{\mathbf{e}}_i^{\mathrm{T}}\, {}^{\mathcal{E}'}\hat{\mathbf{e}}_j\,, \qquad i,\ j = 1,\ 2,\ 3\,.
\end{aligned}
\tag{135}
$$

Noting that the autorepresentation of a basis is the same for every basis,

$$
{}^{\mathcal{E}}\hat{\mathbf{e}}_i^{\mathrm{T}}\, {}^{\mathcal{E}'}\hat{\mathbf{e}}_j = ({}^{\mathcal{E}'}\hat{\mathbf{e}}_i')^{\mathrm{T}}\, {}^{\mathcal{E}'}\hat{\mathbf{e}}_j = \hat{\boldsymbol{e}}_i' \cdot \hat{\boldsymbol{e}}_j\,, \qquad i,\ j = 1,\ 2,\ 3\,,
\tag{136}
$$

and, therefore,

$$
{}^{\mathcal{E}}\hat{\mathbf{e}}_i^{\mathrm{T}}\ \mathsf{A}^{\mathcal{E}'/\mathcal{E}}\ {}^{\mathcal{E}}\hat{\mathbf{e}}_j = \mathsf{A}_{ij}^{\mathcal{E}'/\mathcal{E}}\,, \qquad i,\ j = 1,\ 2,\ 3\,,
\tag{137}
$$

as expected. Likewise, from (134) by similar reasoning

$$
{}^{\mathcal{E}'}\hat{\mathbf{e}}_i^{\mathrm{T}}\ \mathsf{A}^{\mathcal{E}'/\mathcal{E}}\ {}^{\mathcal{E}'}\hat{\mathbf{e}}_j = \mathsf{A}_{ij}^{\mathcal{E}'/\mathcal{E}}\,, \qquad i,\ j = 1,\ 2,\ 3
\tag{138}
$$

and

$$
\begin{aligned}
\mathsf{A}^{\mathcal{E}'/\mathcal{E}} &= \sum_{i=1}^{3}\sum_{j=1}^{3} \mathsf{A}_{ij}^{\mathcal{E}'/\mathcal{E}}\ {}^{\mathcal{E}}\hat{\mathbf{e}}_i\ {}^{\mathcal{E}}\hat{\mathbf{e}}_j^{\mathrm{T}} = \sum_{i=1}^{3}\sum_{j=1}^{3} \mathsf{A}_{ij}^{\mathcal{E}'/\mathcal{E}}\ {}^{\mathcal{E}'}\hat{\mathbf{e}}_i\ {}^{\mathcal{E}'}\hat{\mathbf{e}}_j^{\mathrm{T}} \\
&= \sum_{k=1}^{3} {}^{\mathcal{E}'}\hat{\mathbf{e}}_k\ {}^{\mathcal{E}}\hat{\mathbf{e}}_k^{\mathrm{T}} = \sum_{k=1}^{3} {}^{\mathcal{E}}\hat{\mathbf{e}}_k\ {}^{\mathcal{E}}\hat{\mathbf{e}}_k'^{\mathrm{T}}\,.
\end{aligned}
\tag{139}
$$

The third and fourth expressions in (139) are special cases of (110) and (107), respectively.

The first expression for $\mathsf{A}^{\mathcal{E}'/\mathcal{E}}$ in (139) follows directly from (116) but not the second expression. The two expressions in (139) are reminiscent of the fact that Euler's formula for the attitude matrix may contain either ${}^{\mathcal{E}}\hat{\mathbf{n}}^{\mathcal{E}'/\mathcal{E}}$ or ${}^{\mathcal{E}'}\hat{\mathbf{n}}^{\mathcal{E}'/\mathcal{E}}$.

It follows now from (116), (122), and (139) that the *attitude dyadic* is given by

$$
\mathcal{A}^{\mathcal{E}'/\mathcal{E}} = \sum_{i=1}^{3}\sum_{j=1}^{3} \mathsf{A}_{ij}^{\mathcal{E}'/\mathcal{E}}\ \hat{\boldsymbol{e}}_i\, \hat{\boldsymbol{e}}_j^{\dagger}\,.
\tag{140}
$$

From (140), noting (86), we write

$$
\begin{aligned}
\mathcal{A}^{\mathcal{E}'/\mathcal{E}} &= \sum_{i=1}^{3} \hat{\boldsymbol{e}}_i\ \sum_{j=1}^{3} \mathsf{A}_{ij}^{\mathcal{E}'/\mathcal{E}}\ \hat{\boldsymbol{e}}_j^{\dagger} = \sum_{i=1}^{3} \hat{\boldsymbol{e}}_i \left(\sum_{j=1}^{3} \mathsf{A}_{ij}^{\mathcal{E}'/\mathcal{E}}\ \hat{\boldsymbol{e}}_j\right)^{\dagger} \\
&= \sum_{i=1}^{3} \hat{\boldsymbol{e}}_i\, \hat{\boldsymbol{e}}_i'^{\dagger} = \hat{\boldsymbol{e}}_1 \hat{\boldsymbol{e}}_1'^{\dagger} + \hat{\boldsymbol{e}}_2 \hat{\boldsymbol{e}}_2'^{\dagger} + \hat{\boldsymbol{e}}_3 \hat{\boldsymbol{e}}_3'^{\dagger}\,.
\end{aligned}
\tag{141}
$$

Equation (141) should be compared to the last expression in (139). Thus,

$$\mathcal{A}^{\mathcal{E}'/\mathcal{E}}\,\hat{\boldsymbol{e}}'_k = \hat{\boldsymbol{e}}_k\,, \qquad k = 1,\,2,\,3\,. \tag{142}$$

The attitude dyadic transforms the posterior basis into the prior basis. The attitude matrix transforms the representation of a physical vector with respect to the prior basis into the representation of the same physical vector with respect to the posterior basis.

From (141)

$$\mathcal{A}^{\mathcal{E}''/\mathcal{E}} = \mathcal{A}^{\mathcal{E}'/\mathcal{E}}\mathcal{A}^{\mathcal{E}''/\mathcal{E}'} \tag{143}$$

and

$$\mathcal{A}^{\mathcal{E}'/\mathcal{E}}\big(\mathcal{A}^{\mathcal{E}'/\mathcal{E}}\big)^\dagger = \big(\mathcal{A}^{\mathcal{E}'/\mathcal{E}}\big)^\dagger\mathcal{A}^{\mathcal{E}'/\mathcal{E}} = \mathcal{I}, \tag{144}$$

in analogy with (90). Equation (143) can be compared with the corresponding relationship for the attitude matrix

$$\mathsf{A}^{\mathcal{E}''/\mathcal{E}} = \mathsf{A}^{\mathcal{E}''/\mathcal{E}'}\,\mathsf{A}^{\mathcal{E}'/\mathcal{E}}\,, \tag{145}$$

which follows from (134). The order of multiplication for the composition of attitude dyadics is opposite to that for the attitude matrices.

Comparing Euler's formula for the attitude matrix with that for the attitude dyadic leads to an interesting result. Here, $\hat{\boldsymbol{n}}$ denotes the physical rotation axis and θ is the angle of rotation. For the attitude matrix, we have

$$\mathsf{A}^{\mathcal{E}'/\mathcal{E}} = (\cos\theta)\ \mathsf{I}_{3\times3} + (1-\cos\theta)\,\hat{\mathbf{n}}\hat{\mathbf{n}}^{\mathrm{T}} - (\sin\theta)\,[\,\hat{\mathbf{n}}\times\,]\,, \tag{146}$$

and for the attitude dyadic, correspondingly,

$$\mathcal{A}^{\mathcal{E}'/\mathcal{E}} = (\cos\theta)\,\mathcal{I} + (1-\cos\theta)\,\hat{\boldsymbol{n}}\hat{\boldsymbol{n}}^\dagger - (\sin\theta)\,\{\,\hat{\boldsymbol{n}}\times\,\}\,, \tag{147}$$

Equation (147) follows also from rewriting (75) using the dyadics $\hat{\boldsymbol{n}}\hat{\boldsymbol{n}}^\dagger$ and $\{\hat{\boldsymbol{n}}\times\}$, conjugating that equation to obtain an expression for $\hat{\boldsymbol{e}}'^\dagger_k$, substituting that expression into (141), and noting (131). (Note that there is a sign error in (A20) of [1]).

Note that, once the attitude matrix is defined, the attitude dyadic follows immediately from (122). One cannot define the attitude dyadic independently of the definition of the attitude matrix, for example, by requiring that it transform the prior physical basis into the posterior physical basis, without violating (125).

While the subject of this subsection is the attitude dyadic, the results hold also for the general orthogonal transformation, which may not be described by a proper orthogonal matrix.

Transformation of Physical Quantities and Representations

We now have a complete parallelism between the transformation of physical bases

$$\hat{\boldsymbol{e}}'_k = \mathcal{A}^{\mathcal{E}/\mathcal{E}'}\hat{\boldsymbol{e}}_k\,, \qquad \hat{\boldsymbol{e}}'^\dagger_k = \hat{\boldsymbol{e}}^\dagger_k\left(\mathcal{A}^{\mathcal{E}/\mathcal{E}'}\right)^\dagger\,, \qquad k = 1,\,2,\,3 \tag{148}$$

and the corresponding transformation of basis representations

$${}^{\mathcal{E}'}\hat{\mathbf{e}}_k = \mathsf{A}^{\mathcal{E}'/\mathcal{E}}\ {}^{\mathcal{E}}\hat{\mathbf{e}}_k\,, \qquad {}^{\mathcal{E}'}\hat{\mathbf{e}}^{\mathrm{T}}_k = {}^{\mathcal{E}}\hat{\mathbf{e}}^{\mathrm{T}}_k\ \left(\mathsf{A}^{\mathcal{E}'/\mathcal{E}}\right)^{\mathrm{T}}\,, \qquad k - 1,\,2,\,3\,. \tag{149}$$

Note the differing superscripts on the attitude dyadic and on the attitude matrix.

The corresponding identity operators are

$$\mathcal{I} = \sum_{k=1}^{3} \hat{\boldsymbol{e}}_k \, \hat{\boldsymbol{e}}_k^{\dagger} \,, \qquad \mathsf{I}_{3\times 3} = \sum_{k=1}^{3} {}^{\mathcal{E}}\hat{\mathbf{e}}_k \, {}^{\mathcal{E}}\hat{\mathbf{e}}_k^{\mathrm{T}} \,. \tag{150}$$

The attitude matrix is a function of the representation of the axis of rotation with respect to prior or posterior axes. Thus,

$$\mathsf{A}^{\mathcal{E}'/\mathcal{E}} = \mathsf{A}({}^{\mathcal{E}}\hat{\mathbf{n}}^{\mathcal{E}'/\mathcal{E}}, \, \theta^{\mathcal{E}'/\mathcal{E}}) = \mathsf{A}({}^{\mathcal{E}'}\hat{\mathbf{n}}^{\mathcal{E}'/\mathcal{E}}, \, \theta^{\mathcal{E}'/\mathcal{E}}) \,, \tag{151}$$

where $\mathsf{A}(\,\cdot\,,\,\cdot\,)$ is the function of (146), while for the attitude dyadic, we have simply

$$\mathcal{A}^{\mathcal{E}'/\mathcal{E}} = \mathcal{A}(\hat{\boldsymbol{n}}^{\mathcal{E}'/\mathcal{E}}, \, \theta^{\mathcal{E}'/\mathcal{E}}) \,, \tag{152}$$

where $\mathcal{A}(\,\cdot\,,\,\cdot\,)$ is the function of (147).

There is, however, an important difference between the transformations mediated by the attitude dyadic and the transformations mediated by the attitude matrix. Equations (149) can be applied to any column vector

$${}^{\mathcal{E}'}\mathbf{v} = \mathsf{A}^{\mathcal{E}'/\mathcal{E}} \; {}^{\mathcal{E}}\mathbf{v} \,, \qquad {}^{\mathcal{E}'}\mathbf{v}^{\mathrm{T}} = {}^{\mathcal{E}}\mathbf{v}^{\mathrm{T}} \, \left(\mathsf{A}^{\mathcal{E}'/\mathcal{E}}\right)^{\mathrm{T}} \,. \tag{153}$$

The same is not true for (148). The expression $\mathcal{A}^{\mathcal{E}/\mathcal{E}'} \boldsymbol{v}$ for a physical vector $\boldsymbol{v}$ not closely associated with the basis $\mathcal{E}$ is physically meaningless, because $\mathcal{A}^{\mathcal{E}/\mathcal{E}'}$ is an operator that transforms bases. A case in which $\boldsymbol{v}$ is closely associated with the basis $\mathcal{E}$ is examined below. Likewise, there is no dyadic relationship corresponding to the matrix equation

$${}^{\mathcal{E}'}\mathsf{F} = \mathsf{A}^{\mathcal{E}'/\mathcal{E}} \; {}^{\mathcal{E}}\mathsf{F} \, \left(\mathsf{A}^{\mathcal{E}'/\mathcal{E}}\right)^{\mathrm{T}} \,, \tag{154}$$

because the dyadic $\mathcal{F}$ is frame independent. The attitude dyadic is useful only for transforming physical bases and is not needed for this operation, because we can write with greater ease than (148)

$$\hat{\boldsymbol{e}}_i' = \sum_{j=1}^{3} A_{ij}^{\mathcal{E}'/\mathcal{E}} \, \hat{\boldsymbol{e}}_j \,, \qquad \hat{\boldsymbol{e}}_i'^{\dagger} = \sum_{j=1}^{3} A_{ij}^{\mathcal{E}'/\mathcal{E}} \, \hat{\boldsymbol{e}}_j^{\dagger} \,, \qquad i = 1,\, 2,\, 3\,. \tag{155}$$

To see how the expression $\mathcal{A}^{\mathcal{E}/\mathcal{E}'} \boldsymbol{v}$ can be meaningful when $\boldsymbol{v}$ is closely associated with the basis $\mathcal{E}$, consider a physical vector $\boldsymbol{v}$, which we write in the usual manner with respect to a right-handed orthonormal basis $\mathcal{E}$ as

$$\boldsymbol{v} = {}^{\mathcal{E}}v_1 \, \hat{\boldsymbol{e}}_1 + {}^{\mathcal{E}}v_2 \, \hat{\boldsymbol{e}}_2 + {}^{\mathcal{E}}v_3 \, \hat{\boldsymbol{e}}_3 \,, \tag{156}$$

and therefore,

$$\mathcal{A}^{\mathcal{E}/\mathcal{E}'} \boldsymbol{v} = {}^{\mathcal{E}}v_1 \, \hat{\boldsymbol{e}}_1' + {}^{\mathcal{E}}v_2 \, \hat{\boldsymbol{e}}_2' + {}^{\mathcal{E}}v_3 \, \hat{\boldsymbol{e}}_3' \,. \tag{157}$$

$\mathcal{A}^{\mathcal{E}/\mathcal{E}'}\boldsymbol{v}$ has the same components with respect to $\mathcal{E}'$ as $\boldsymbol{v}$ has with respect to $\mathcal{E}$. Let us suppose now that the basis $\mathcal{E}$ is a time varying basis $\mathcal{E}(t)$, and that $\boldsymbol{v}$ is a time varying physical vector $\boldsymbol{v}(t)$, fixed and constant with respect to $\mathcal{E}(t)$, that is, ${}^{\mathcal{E}(t)}\mathbf{v}(t)$ is constant in time. For example, $\boldsymbol{v}(t)$ might be a sensor boresight fixed in a rotating rigid spacecraft with body axes $\mathcal{E}(t)$. Then $\boldsymbol{v}(t') = \mathcal{A}^{\mathcal{E}(t)/\mathcal{E}(t')}\boldsymbol{v}(t)$. This example is not very different from the transformation of physical basis vectors. Since we ultimately need the transformed column vector rather than the transformed physical vector, the application of the attitude matrix in (153) has wider application than (148).

Vectrices

The above formalism makes possible a more intuitive expression for the vectrix [3, 17, 18]. The *vectrix* $\mathcal{V}^{E/\mathcal{E}}$ mediates the transformation $\mathcal{V}^{E/\mathcal{E}} : \boldsymbol{u} \to {}^{\mathcal{E}}\mathbf{u}$ according to

$$ {}^{\mathcal{E}}\mathbf{u} = \mathcal{V}^{E/\mathcal{E}}\boldsymbol{u} \,. \tag{158} $$

In (158) we use E in the superscript of the vectrix as a shorthand for ${}^{\mathcal{E}}\mathcal{E}$. Note the different typefaces in ${}^{\mathcal{E}}\mathcal{V}$ and $\mathcal{V}^{E/\mathcal{E}}$, as well as the different superscripts.

The vectrix $\mathcal{V}^{E/\mathcal{E}}$ can be written as the bilinear form

$$ \mathcal{V}^{E/\mathcal{E}} = \sum_{i=1}^{3} {}^{\mathcal{E}}\hat{\mathbf{e}}_i\, \hat{\boldsymbol{e}}_i^{\dagger} \,. \tag{159} $$

The vectrix is thus intermediate between a dyadic and a matrix. We define the *conjugate vectrix* $\left(\mathcal{V}^{E/\mathcal{E}}\right)^{\dagger}$ as

$$ \left(\mathcal{V}^{E/\mathcal{E}}\right)^{\dagger} = \mathcal{V}^{\mathcal{E}/E} = \sum_{i=1}^{3} \hat{\boldsymbol{e}}_i\, {}^{\mathcal{E}}\hat{\mathbf{e}}_i^{\mathrm{T}} \,. \tag{160} $$

The conjugate vectrix satisfies

$$ \boldsymbol{u} = \left(\mathcal{V}^{E/\mathcal{E}}\right)^{\dagger} {}^{\mathcal{E}}\mathbf{u} = \mathcal{V}^{\mathcal{E}/E}\, {}^{\mathcal{E}}\mathbf{u} \,. \tag{161} $$

The four bilinear transformation operators are summarized in Table 1. The columns correspond to the domain of the operator, either $\mathcal{V}$ or ${}^{\mathcal{E}}\mathcal{V}$, while the rows correspond by the image space.

For $\mathcal{E}$ an orthonormal basis,

$$ \mathcal{F} = \mathcal{V}^{\mathcal{E}/E}\, {}^{\mathcal{E}}\mathsf{F}\, \mathcal{V}^{E/\mathcal{E}} = \left(\mathcal{V}^{E/\mathcal{E}}\right)^{\dagger} {}^{\mathcal{E}}\mathsf{F}\, \mathcal{V}^{E/\mathcal{E}} \,, \tag{162} $$

and

$$ {}^{\mathcal{E}}\mathsf{F} = \mathcal{V}^{E/\mathcal{E}}\, \mathcal{F}\, \mathcal{V}^{\mathcal{E}/E} = \mathcal{V}^{E/\mathcal{E}}\, {}^{\mathcal{E}}\mathcal{F} \left(\mathcal{V}^{E/\mathcal{E}}\right)^{\dagger} . \tag{163} $$

We have also, for mixed bases,

$$ \mathcal{V}^{E'/\mathcal{E}'} \left(\mathcal{V}^{E/\mathcal{E}}\right)^{\dagger} = \mathsf{A}^{\mathcal{E}'/\mathcal{E}} \tag{164} $$

$\cdot$	$\mathcal{V}$	${}^{\mathcal{E}}\mathcal{V}$
$\mathcal{V}$	Attitude Dyadic	$(\text{Vectrix})^{\dagger}$
${}^{\mathcal{E}}\mathcal{V}$	Vectrix	Attitude Matrix

Table 1. Bilinear transformation operators on vector spaces. The columns correspond to the domain of the operator, either $\mathcal{V}$ or ${}^{\mathcal{E}}\mathcal{V}$. while the rows correspond to the image space. $\mathcal{V}$ is the physical vector space, and ${}^{\mathcal{E}}\mathcal{V}$ is the vector space of column-vector representations with respect to the basis $\mathcal{E}$.

and

$$\left(\mathcal{V}^{E/\mathcal{E}}\right)^{\dagger}\mathcal{V}^{E'/\mathcal{E}'} = \mathcal{A}^{\mathcal{E}'/\mathcal{E}} \tag{165}$$

Note that $E = {}^{\mathcal{E}}\mathcal{E}$ and $E' = {}^{\mathcal{E}'}\mathcal{E}'$ are the identical column-vector basis.

The statement about the utility of the attitude dyadic applies also to the vectrix. Despite the prominence of vectrices in [3], vectrices are not of practical use.

A World without Physical Vectors

What happens in a world without physical vectors, where the only concept of vectors is as a collections of components, u_i, $i = 1, 2, 3$, which the ancients did not usually write as indexed components, but more frequently as a, b, c. All of the relationsships for physical vectors are no longer at our disposal, and vectors. We must recall that vectors were not introduced until the 1890s and matrix algebra was not invented until the 1850s, so that vectors were not even written as 3×1 arrays. The first vehicles for writing vectors even as a single characteor ws Hamilton's quaternions in 1844.

Supposing that one did write vectors as a 3×1 array of components with respect to a basis, and we do have vector and matrix algebra at our disposal, a change of bais ob a vector

Summary and Discussion

This work has presented the development of physical vectors, column vectors, row vectors, dual vectors, conjugate dual vectors, the attitude matrix, the attitude dyadic, the vectrix, and the conjugate vectrix. The physical vectors are the vectors of diagrams and are indispensable, therefore, for formulating problems. The column-vector representation of a physical vector is the vector of a measurement and also the vector figuring in numerical computations. We cannot do without either.

This article developed also the axis-angle representation of rotations $(\hat{\mathbf{n}}, \theta)$ and the rotation-vector representation $\boldsymbol{\theta}$. The latter is important in the description of attitude errors and corrections. The attitude matrix can be written [1] as

$$\mathsf{A}^{\mathcal{E}'/\mathcal{E}} = \exp\{-[\boldsymbol{\theta}^{\mathcal{E}'/\mathcal{E}}\times]\} \,, \tag{166}$$

where $\exp\{\,\cdot\,\}$ is the matrix exponential function [19]. Thus, the study of attitude becomes in a sense the study of the antisymmetric matrix $[\mathbf{u}\times]$ or, even more austerely, the study of the Levi-Civita symbol. To stretch the point even further, we might say that the study of attitude is the study of the vector product. Further details of the axis-angle and rotation-vector representations, as well as many other attitude representations, can be found in [1].

The only use for the attitude dyadic is the transformation of physical basis vectors, which can be carried out with greater ease using the attitude matrix, as in (155), than with the attitude dyadic, as in (148). Other dyadics, such as the inertia dyadic, are useful, because they permit us to formulate equations of motion for physical vectors in a coordinate-free manner. Dyadics are useful, therefore, in extending the present formalism to attitude dynamics. Having a parallelism between the attitude matrix and the attitude dyadic is satisfying intellectually, but this parallelism, as we have seen, has little practical value, as demonstrated in the subsection "Transformation of Physical Quantities and Representations." We have presented the attitude dyadic, however, not just to "complete the math," but also because the attitude dyadic and the attitude matrix are often confused in approaches that do not distinguish between physical vectors and their representations as column vectors. Some greater clarity is needed to dispel this confusion.

Gibbs [15] gave his version of Euler's formula not in the form of (74) above but in terms of dyadics. In 1901, matrices were not used to the same degree as they are today.

The diagram seen frequently in mission support, depicting "reference" vectors (representations with respect to the inertial frame) and "observation" vectors (representations with respect to the body frame) on a single picture connected by the attitude matrix is inherently wrong. Depicted vectors are physical vectors and frame independent. The attitude matrix finds no place in such a diagram. On the other hand, diagrams depicting vectors fixed in a rotating body at different times related by a matrix for the relative attitude between those two times are perfectly correct. The two cases are intrinsically different. A failure to honor that difference can be (and has been) the cause of sign errors in mission software.

We can, of course, make a picture of the prior and the posterior representations of physical vectors using the entries of each column vector to plot the representation. Consider the case of a single physical vector $\boldsymbol{v}$ represented with respect to the inertial frame (basis $\mathcal{I}$, which we identify with $\mathcal{E}$) and with respect to the body frame (basis $\mathcal{B}$, which we identify with $\mathcal{E}'$). Let us suppose that $\mathcal{E}$ and $\mathcal{E}'$ are the same as in Figure 3, and that the physical vector $\boldsymbol{v}$ lies in the plane of $\hat{\boldsymbol{e}}_1$ and $\hat{\boldsymbol{e}}_2$. Then a picture of ${}^{\mathcal{I}}\mathbf{v}$ and ${}^{\mathcal{B}}\mathbf{v}$ might look like Figure 4. The physical rotation of the physical inertial basis ($\mathcal{I}$, which is $\mathcal{E}$) into the physical body basis ($\mathcal{B}$, which is $\mathcal{E}'$) in Figure 3 is clockwise by an angle θ, while the "rotation" of the column vector ${}^{\mathcal{I}}\mathbf{v}$ into the column vector ${}^{\mathcal{B}}\mathbf{v}$ in Figure 4 is counterclockwise by the same angle θ. Note that the axes of the picture in Figure 4 are labeled x and y, because such axes are not physical axes. The x- and y-axes of the picture cannot correspond to directions in real space, because ${}^{\mathcal{I}}\mathbf{v}$ and ${}^{\mathcal{B}}\mathbf{v}$ are not directions in real space. The representations ${}^{\mathcal{I}}\mathbf{v}$ and ${}^{\mathcal{B}}\mathbf{v}$ correspond to the *same* physical direction in real space. We could, of course, draw the inertially referenced column vectors on one picture and the body-referenced column vectors on a second picture and connect the two pictures by the attitude matrix. Such a pair of pictures, as illustrated in Figure 5, is different from the diagrams of Figures 1 and 3. Obviously, we must be careful not to confuse such pictures with physical diagrams.

The important lessons of this tutorial are that physical vectors are not column vectors, and consequently, that dyadics are not matrices. Although physical vectors are very useful in attitude studies, the attitude dyadic really is less so, and therefore, it is advantageous to convert a physical-vector formulation to a column-vectors formulation early. Also, things are not always what they seem to be at first glance. The failure of the autorepresentation of a basis (58)–(60) to be necessarily an orthonormal column-vector basis despite appearances is certainly one case in point. Column vectors are not vectors in space, and we can get into trouble if we picture column vectors in a manner appropriate to physical vectors.

The present tutorial is an expansion and modification of parts of an earlier work [1]. That work remains valid, but the present treatment is more complete and more transparent for the subset of [1] that is covered here. The goal of the present article has not yet been fully attained. The task remains to develop attitude kinematics and dynamics within the same framework and with the same attention to detail.

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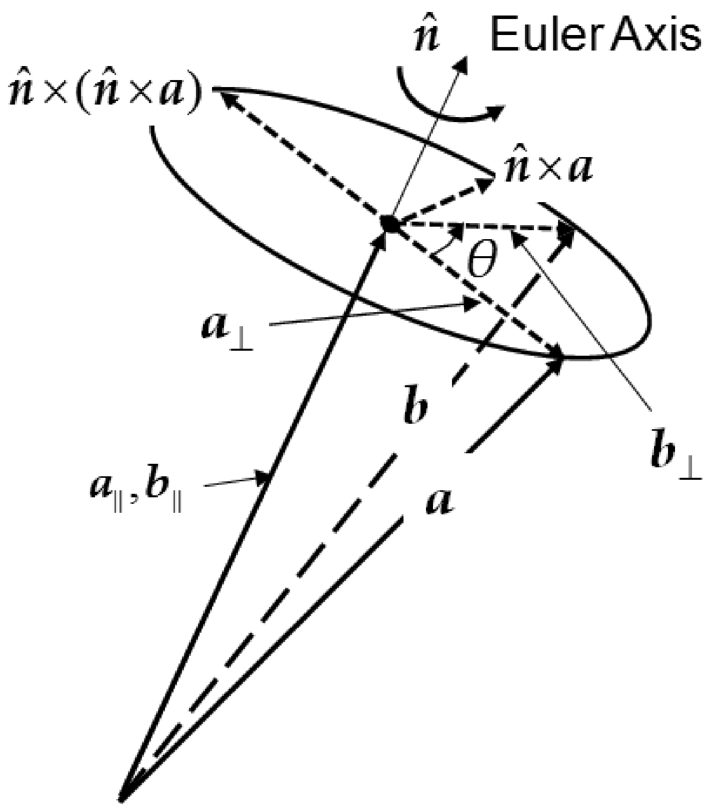

Figure 1. A physical rotation in three dimensions. θ is the angle of rotation, and $\hat{\boldsymbol{n}}$ the physical axis of rotation. The prior physical vector is $\boldsymbol{a}$, and the posterior physical vector is $\boldsymbol{b}$. The subscripts $\parallel$ and $\perp$ denote components parallel and perpendicular to $\hat{\boldsymbol{n}}$, respectively.

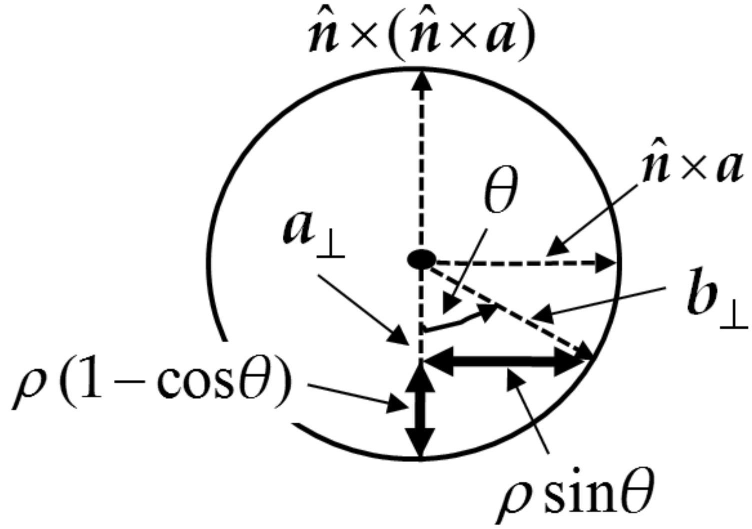

Figure 2. A physical rotation as seen in the tangent plane. θ is the angle of rotation, and $\hat{\boldsymbol{n}}$ the physical axis of rotation. The prior physical vector is $\boldsymbol{a}$, and the posterior physical vector is $\boldsymbol{b}$. The subscripts $\parallel$ and $\perp$ denote components parallel and perpendicular to $\hat{\boldsymbol{n}}$, respectively. The quantity ρ is equal to $|\hat{\boldsymbol{n}}\times\boldsymbol{a}|$.

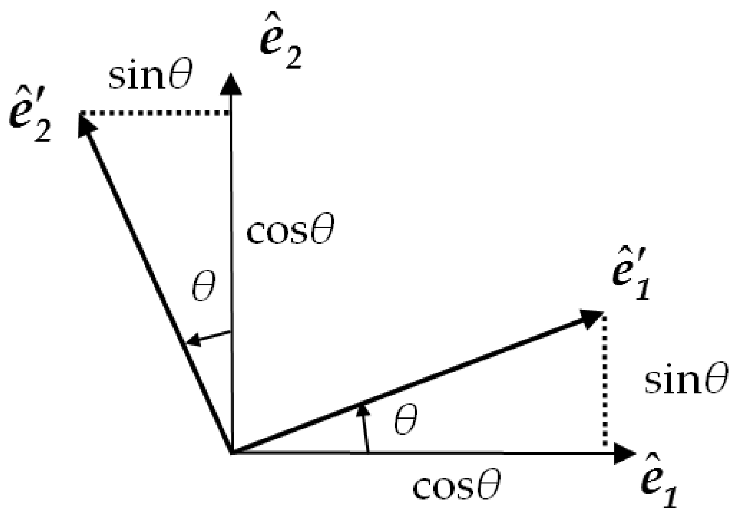

Figure 3. A rotation in the plane. The physical vectors $\hat{e}_1$ and $\hat{e}_2$ are the prior basis, and the physical vectors $\hat{e}'_1$ and $\hat{e}'_2$ are the posterior basis. θ is the angle of rotation, and the physical axis of rotation is $\hat{e}_3$, which is the same as $\hat{e}'_3$ (neither shown).

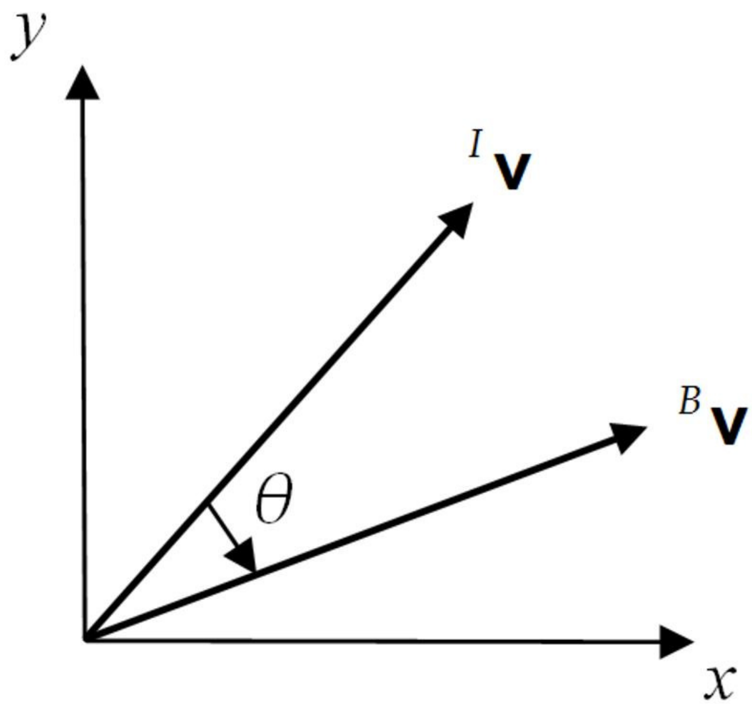

Figure 4. Two column-vector representations. ${}^{I}\mathbf{v}$ and ${}^{B}\mathbf{v}$ are the representations of the physical vector $\boldsymbol{v}$ with respect to the inertial and body frames, respectively. The basis of the inertial frame is the same as $\mathcal{E}$ in Figure 3, while the basis of the body frame is the same as $\mathcal{E}$' in Figure 3.

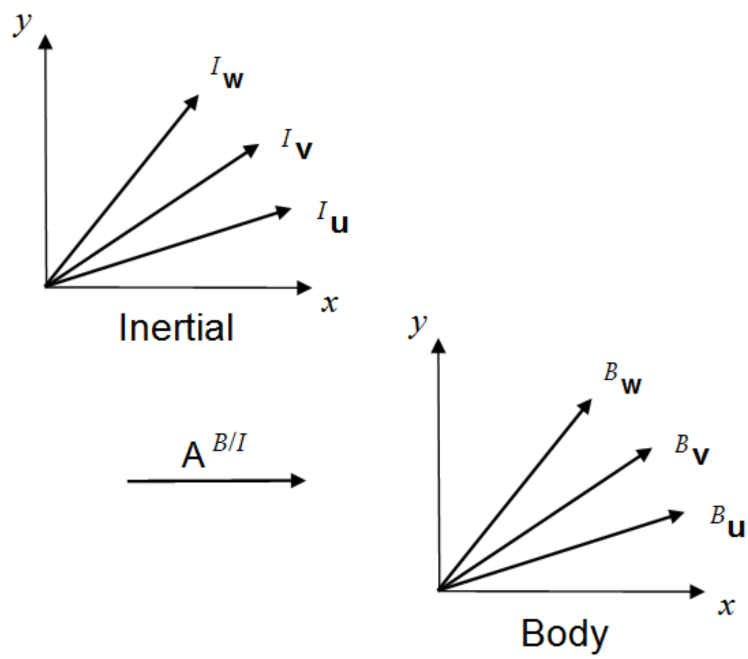

Figure 5. Transformation of inertial representations to body representations. The upper diagram shows the inertial representations of the physical vectors $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$, while the lower diagram shows the body representations of the same physical vectors. $A^{B/I}$ is the attitude matrix that transforms column-vector representations from the inertial frame to the body frame.