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Effective Direction Measurements for Spacecraft Attitude: III. Defective Directions and Data Fusion

Malcolm D. Shuster[1]

Yet better the excess than the defect.

Henry Wadsworth Longfellow (1807–1882)
Morituri Salutamus (1875)

Abstract

Two effective measurements are examined which represent attitude information defectively, that is, part of the information is discarded in the effective measurement. These effective measurements are tested for the fusion of a star-tracker attitude estimate with a direction measurement from a vector Sun sensor. The defective directions tested are the Brozenec-Bender vector and those in the prescription of Bar-Itzhack and Harman. These are compared with maximum-likelihood estimation based on all measurements and with the star-tracker attitude estimate alone. Naturally, the approximate algorithms do not perform as well as maximum-likelihood estimation without approximation.

Introduction: Effective Measurements

In the first two parts of this work [1, 2] we examined two effective direction measurements: the equivalent direction measurements [1] and the predicted direction measurements [2]. The *three* equivalent directions or the *two* predicted directions, together with their covariance matrices, contained all of the attitude information in a given arbitrary set of attitude measurements from which the attitude was observable. On the basis of the three equivalent or two predicted directions, the application of maximum-likelihood estimation would lead to the identical attitude estimate and attitude covariance matrix as from the original measurement set. The

[1]Director of Research, Acme Spacecraft Company, 13017 Wisteria Drive, Box 328, Germantown, Maryland 20874. E-mail: mdshuster@comcast.net. Website: http://home.comcast.net/~mdshuster.

avowed purpose of such effective measurements was to provide the original attitude information in a compact form for later applications.

The equivalent directions and the predicted directions are not the only effective measurements for the attitude. The estimate of the attitude matrix (direction-cosine matrix) also provides an effective measurement, and an algorithm for optimally combining independent estimates of the attitude matrix was examined in reference [3] (and presented again here) and was shown to reduce to a generalization of the Wahba problem [3–7].[2] The attitude profile matrix B and the Davenport matrix K also provide a complete representation of the attitude information (including the attitude covariance matrix) and may be used in data fusion. This was the point of reference [3].

In the present work, we examine two further effective measurements which can reproduce only approximately a given attitude estimate and attitude covariance matrix. These are the Brozenec-Bender vector [10] and the effective measurement vectors of the prescription of Bar-Itzhack and Harman [11].

Data Fusion within Maximum-Likelihood Estimation

Preamble: The Measurements

There are many ways to mechanize data fusion for our simple problem within maximum-likelihood estimation, depending on how one wishes to represent the data. We assume that the original star-tracker measurements were of the form

$$\mathbf{z}_k^{\text{star}} = \mathbf{f}_k(C_{\text{ST}}^{\text{true}}) + \Delta\mathbf{z}_k^{\text{star}}, \qquad k = 1, \ldots, N \tag{1a}$$

$$\Delta\mathbf{z}_k^{\text{star}} \sim \mathcal{N}(\mathbf{0}, R_k^{\text{star}}), \qquad k = 1, \ldots, N \tag{1b}$$

As pointed out in references [1] and [2], this is not a necessary assumption, but will prove convenient for our calculations in the present work. Here, following the notation of reference [3], C_{ST} denotes the attitude based on star-tracker measurements only, and A will denote the attitude based on all measurements. Clearly,

$$C_{\text{ST}}^{\text{true}} = A^{\text{true}} \tag{2}$$

which assumes that the axes of the star tracker are also the body axes of the spacecraft. We assume that the $\Delta\mathbf{z}_k^{\text{star}}$, $k = 1, \ldots, N$, are independent, white, Gaussian, zero-mean, and with covariance matrix R_k, $k = 1, \ldots, N$. We assume that there are always sufficient measurements for the attitude to be observable. In general, we follow the notational conventions of reference [12].

Given our assumptions about the measurements, the optimal star-tracker attitude estimate is the value $C_{\text{ST}}^{*\prime}$ which minimizes the cost function

$$J(C_{\text{ST}}) = \frac{1}{2}\sum_{k=1}^{N}(\mathbf{z}_k^{\text{star}\prime} - \mathbf{f}_k(C_{\text{ST}}))^{\text{T}}(R_k^{\text{star}})^{-1}(\mathbf{z}_k^{\text{star}\prime} - \mathbf{f}_k(C_{\text{ST}})) \tag{3}$$

General solution methods for cost functions of this form have been examined in detail in reference [13]. The covariance matrix of C_{ST}^{*}, the star-tracker attitude *estimator,* is R_{ST}, the covariance matrix of $\tilde{\boldsymbol{\epsilon}}^{*}$, the star-tracker attitude estimator error, is defined by

[2]Reference [7] provides a masterful overview of the many solutions to the Wahba problem. On its numerical results and their interpretation, see references [8] and [9].

$$C_{\mathrm{ST}}^{*} = e^{[[\tilde{\boldsymbol{\epsilon}}^{*}]]} C_{\mathrm{ST}}^{\mathrm{true}} \tag{4}$$

where $[[\mathbf{u}]]$ is defined as[3]

$$[[\mathbf{u}]] \equiv \begin{bmatrix} 0 & u_3 & -u_2 \\ -u_3 & 0 & u_1 \\ u_2 & -u_2 & 0 \end{bmatrix} \tag{5}$$

The random attitude error vector $\tilde{\boldsymbol{\epsilon}}^{*}$ will not be Gaussian in general. However, if the measurement functions $\mathbf{f}_k$ are linear in C_{ST}, and the number of measurements is finite, then to lowest nonvanishing order in $\tilde{\boldsymbol{\epsilon}}^{*}$

$$\tilde{\boldsymbol{\epsilon}}^{*} \sim \mathcal{N}(\mathbf{0}, R_{\mathrm{ST}}) \tag{6}$$

Equation (4) as a measurement equation is very different from equation (1a). The measurement noise is multiplicative rather than additive. Had we wished to present the attitude error as additive effective measurement noise, we could have written[4]

$$C_{\mathrm{ST}}^{*} = C_{\mathrm{ST}}^{\mathrm{true}} + \Delta C_{\mathrm{ST}}^{*} \tag{7}$$

with

$$\Delta C_{\mathrm{ST}}^{*} = [[\tilde{\boldsymbol{\epsilon}}^{*}]]\, C_{\mathrm{ST}}^{\mathrm{true}} \tag{8}$$

which is reminiscent of the errors in the predicted directions [2]. It is far more convenient in general to work in terms of $\tilde{\boldsymbol{\epsilon}}^{*}$ rather than in terms of $\Delta C_{\mathrm{ST}}^{*}$ because of the lower dimension.[5]

The Sun-sensor measurement will be assumed to be characterized by additive zero-mean Gaussian measurement noise independent of the star measurements

$$\mathbf{z}_S = \mathbf{f}_S(A^{\mathrm{true}}) + \Delta\mathbf{z}_S \tag{9a}$$

$$\Delta\mathbf{z}_S \sim \mathcal{N}(\mathbf{0}, R_S) \tag{9b}$$

and $\mathbf{z}_S$ will be of dimension two and, therefore, R_S will be of full rank.

We can write equations (9) equivalently as

$$\hat{\mathbf{W}}_S = A^{\mathrm{true}}\hat{\mathbf{V}}_S + \Delta\hat{\mathbf{W}}_S \equiv \hat{\mathbf{W}}_S^{\mathrm{true}} + \Delta\hat{\mathbf{W}}_S \tag{10a}$$

$$\Delta\hat{\mathbf{W}}_S \sim \mathcal{N}(\mathbf{0}, R_{S'}) \tag{10b}$$

$$R_{S'} = \mathsf{U}_S R_S \mathsf{U}_S^{\mathrm{T}} \tag{10c}$$

where U_S is a 3×2 matrix satisfying

$$\mathsf{U}_S^{\mathrm{T}}\mathsf{U}_S = I_{2\times 2} \qquad \text{and} \qquad \mathsf{U}_S\mathsf{U}_S^{\mathrm{T}} = I_{3\times 3} - \hat{\mathbf{W}}_S^{\mathrm{true}}\hat{\mathbf{W}}_S^{\mathrm{true}\,\mathrm{T}} \tag{11}$$

For the special case

$$R_S = \sigma_S^2 I_{2\times 2} \tag{12}$$

[3]Some authors prefer to use $[\mathbf{u}\times] \equiv -[[\mathbf{u}]]$.
[4]To first order in $\tilde{\boldsymbol{\epsilon}}^{*}$, which will be characteristic of most relationships in this work.
[5]Reference [5], in the derivation of the TRIAD attitude covariance matrix, contradicts this assertion, but it is true here.

we obtain to lowest order[6]

$$R_{S'} = \sigma_S^2 (I_{3\times 3} - \hat{\mathbf{W}}_S^{\text{true}} \hat{\mathbf{W}}_S^{\text{true}\,\mathrm{T}}) \tag{13}$$

and the Sun-sensor measurement model in this case is QUEST-like, that is, it conforms to the QUEST measurement model [5, 6].

Data Fusion using the Estimate of the Attitude Matrix Directly

From equation (31) of reference [3] we may write the combined cost function based on the star-tracker attitude estimate and the Sun measurement as

$$J(A) = \frac{1}{2}\,\mathrm{tr}\big[(C_{\mathrm{ST}}^{*\prime} - A)^{\mathrm{T}} D_{\mathrm{ST}} (C_{\mathrm{ST}}^{*\prime} - A)\big] + \frac{1}{2}(\mathbf{z}_S' - \mathbf{f}_S(A))^{\mathrm{T}} R_S^{-1} (\mathbf{z}_S' - \mathbf{f}_S(A)) \tag{14}$$

The minimizing value of A for this cost function is the optimal estimate $A^{*\prime}$. Here, D_{ST}, the attitude co-information matrix [1, 3], is given by

$$D_{\mathrm{ST}} = \left(\frac{1}{2}\,\mathrm{tr}\, R_{\mathrm{ST}}^{-1}\right) I_{3\times 3} - R_{\mathrm{ST}}^{-1} \tag{15}$$

If A_{o} is an attitude matrix infinitesimally close to A^{true}, then we may define a new 3×1 variable $\boldsymbol{\xi}$ and its estimate $\boldsymbol{\xi}_{\mathrm{ST}}^{*\prime}$ defined by

$$A \equiv A(\boldsymbol{\xi}) = e^{[[\boldsymbol{\xi}]]} A_{\mathrm{o}} \qquad \text{and} \qquad C_{\mathrm{ST}}^{*\prime} \equiv e^{[[\boldsymbol{\xi}_{\mathrm{st}}^{*\prime}]]} A_{\mathrm{o}} \tag{16ab}$$

then equation (14) becomes

$$\begin{aligned} J(A) &= \frac{1}{2}(\boldsymbol{\xi}_{\mathrm{ST}}^{*\prime} - \boldsymbol{\xi})^{\mathrm{T}} R_{\mathrm{ST}}^{-1} (\boldsymbol{\xi}_{\mathrm{ST}}^{*\prime} - \boldsymbol{\xi}) + \frac{1}{2}(\mathbf{z}_S' - \mathbf{f}_S(A(\boldsymbol{\xi})))^{\mathrm{T}} R_S^{-1} (\mathbf{z}_S' - \mathbf{f}_S(A(\boldsymbol{\xi}))) \\ &= \frac{1}{2}(\boldsymbol{\xi}_{\mathrm{ST}}^{*\prime} - \boldsymbol{\xi})^{\mathrm{T}} R_{\mathrm{ST}}^{-1} (\boldsymbol{\xi}_{\mathrm{ST}}^{*\prime} - \boldsymbol{\xi}) \\ &\quad + \frac{1}{2}(\mathbf{z}_S' - \mathbf{f}_S(A_{\mathrm{o}}) - H_S^{\mathrm{o}}\boldsymbol{\xi})^{\mathrm{T}} R_S^{-1} (\mathbf{z}_S' - \mathbf{f}_S(A_{\mathrm{o}}) - H_S^{\mathrm{o}}\boldsymbol{\xi}) \end{aligned} \tag{17}$$

for some H_S^{o} [13]. We may write equation (14) (or (17)) equivalently as

$$J(A) = \frac{1}{2}\,\mathrm{tr}\big[(C_{\mathrm{ST}}^{*\prime} - A)^{\mathrm{T}} D_{\mathrm{ST}} (C_{\mathrm{ST}}^{*\prime} - A)\big] + \frac{1}{2}(\hat{\mathbf{W}}_S' - A\hat{\mathbf{V}})^{\mathrm{T}} R_{S'}^{\#} (\hat{\mathbf{W}}_S' - A\hat{\mathbf{V}}) \tag{18}$$

where $R_{S'}^{\#}$ is the Moore-Penrose pseudo-inverse of $R_{S'}$, given by

$$R_{S'}^{\#} = \mathsf{U}_S R_S^{-1} \mathsf{U}_S^{\mathrm{T}} \tag{19}$$

which is more interesting formally than computationally. However, when $\hat{\mathbf{W}}_S$ is QUEST-like, equation (18) becomes the Wahba cost function.

The attitude information matrix taking account of both the star-tracker estimate and the Sun-sensor measurement is given by

$$P_{\mathrm{ST}}^{-1} = R_{\mathrm{ST}}^{-1} + R_{S'}^{\#} \tag{20}$$

Equation (17), which, effectively, is used in the attitude Kalman filter [15], was first presented in essentially this form in references [16] and [17].

[6]Cheng, Crassidis and Markley [14] have examined less approximate forms that that of equation (13) and found a 30 percent improvement in attitude estimation accuracy, but at the cost of a much greater computational burden.

Data Fusion Using Equivalent Directions

The covariance matrix for the star-tracker attitude may be written in spectral decomposition in the form

$$R_{\rm ST} = \sum_{i=1}^{3} \tau_i^2 \hat{\mathbf{u}}_i \hat{\mathbf{u}}_i^{\rm T} \tag{21}$$

where $\hat{\mathbf{u}}_k$, $i = 1, 2, 3$, are the characteristic vectors (eigenvectors) of the attitude covariance matrix and τ_i^2, $i = 1, 2, 3$, are the characteristic variances (principal variances, eigenvariances). The equivalent direction measurements, equivalent reference directions, and equivalent variances are then given by reference [1]

$$\hat{\mathbf{W}}_i^{\rm eq\prime} = \hat{\mathbf{u}}_i, \qquad \hat{\mathbf{V}}_i^{\rm eq} = C_{\rm ST}^{*\prime{\rm T}} \hat{\mathbf{W}}_i^{\rm eq\prime}, \qquad i = 1, 2, 3 \tag{22ab}$$

$$\frac{1}{(\sigma_i^2)^{\rm eq}} = \frac{1}{2}\left(\frac{1}{\tau_1^2} + \frac{1}{\tau_2^2} + \frac{1}{\tau_3^2}\right) - \frac{1}{\tau_i^2}, \qquad i = 1, 2, 3 \tag{22c}$$

By explicit construction [1], the equivalent direction measurements are QUEST-like.

$$\hat{\mathbf{W}}_i^{\rm eq} = C_{\rm ST}^{\rm eq\ true} \hat{\mathbf{V}}_i^{\rm eq} + \Delta\hat{\mathbf{W}}_i^{\rm eq} = \hat{\mathbf{W}}_i^{\rm eq\ true} + \Delta\hat{\mathbf{W}}_i^{\rm eq}, \qquad i = 1, 2, 3 \tag{23a}$$

$$\Delta\hat{\mathbf{W}}_i^{\rm eq} \sim \mathcal{N}(\mathbf{0}, R_i^{\rm eq}), \qquad i = 1, 2, 3 \tag{23b}$$

$$E\{\Delta\hat{\mathbf{W}}_i^{\rm eq} \Delta\hat{\mathbf{W}}_j^{\rm eq T}\} = \delta_{ij} R_i^{\rm eq} \qquad i, j = 1, 2, 3 \tag{23c}$$

with

$$C_{\rm ST}^{\rm eq\ true} = C_{\rm ST}^{*\prime} \tag{23d}$$

$$R_i^{\rm eq} = (\sigma_i^2)^{\rm eq} (I_{3\times 3} - \hat{\mathbf{W}}_i^{\rm eq\ true} \hat{\mathbf{W}}_i^{\rm eq\ true T}), \qquad i = 1, 2, 3 \tag{23e}$$

In terms of the equivalent directions, the cost function for A becomes

$$J(A) = \frac{1}{2} \sum_{i=1}^{3} \frac{1}{(\sigma_i^2)^{\rm eq}} |\hat{\mathbf{W}}_i^{\rm eq\prime} - A\hat{\mathbf{V}}_i^{\rm eq}|^2 + \frac{1}{2}(\hat{\mathbf{W}}_S' - A\hat{\mathbf{V}}_S)^{\rm T} R_{S'}^{\#} (\hat{\mathbf{W}}_S' - A\hat{\mathbf{V}}_S) \tag{24}$$

Clearly, equation (20) holds equally well when the data fusion is mechanized by means of the equivalent directions.

Data Fusion with Predicted Measurements

The predicted direction measurements [2] are given by

$$\hat{\mathbf{W}}_i^{\rm pred} = C_{\rm ST}^* \hat{\mathbf{V}}_i^{\rm pred} \equiv C_{\rm ST}^{\rm true} \hat{\mathbf{V}}_i^{\rm pred} + \Delta\hat{\mathbf{W}}_i^{\rm pred} \equiv \hat{\mathbf{W}}_i^{\rm true} + \Delta\hat{\mathbf{W}}_i^{\rm pred}, \qquad i = 1, 2 \tag{25}$$

Comparing equations (25) and (4) we must have

$$\Delta\hat{\mathbf{W}}_i^{\rm pred} = -[[\hat{\mathbf{W}}_i^{\rm true}]]\tilde{\boldsymbol{\epsilon}}^*, \qquad i = 1, 2 \tag{26a}$$

$$E\{\Delta\hat{\mathbf{W}}_i^{\rm pred} \Delta\hat{\mathbf{W}}_j^{\rm pred T}\} = [[\hat{\mathbf{W}}_i^{\rm true}]] R_{\rm ST} [[\hat{\mathbf{W}}_j^{\rm true}]]^{\rm T} \qquad i, j = 1, 2 \tag{26b}$$

so that the errors of the predicted directions are Gaussian and zero-mean but are also correlated. Clearly, they are not QUEST-like.

To implement the predicted directions in data fusion, we must again select an $A_{\rm o}$ which we expect to be infinitesimally close to $A^{\rm true}$ and define

$$\hat{\mathbf{W}}_i^{\rm o} = A_{\rm o} \hat{\mathbf{V}}_i^{\rm pred}, \qquad i = 1, 2 \tag{27}$$

and then, because $\hat{\mathbf{W}}_1^{\rm pred}$ and $\hat{\mathbf{W}}_2^{\rm pred}$ are not statistically independent, we must define a 3×1 measurement $\mathbf{Z}^{\rm pred\prime}$ according to

$$\mathbf{z}'_{\mathrm{ST}} = H_{\mathrm{ST}}^{\mathrm{oT}} \begin{bmatrix} -(\hat{\mathbf{W}}_1^{\mathrm{o}} \times \hat{\mathbf{W}}_2^{\mathrm{o}}) \cdot \hat{\mathbf{W}}_2^{\mathrm{pred}\,\prime} \\ -(\hat{\mathbf{W}}_1^{\mathrm{o}} \times \hat{\mathbf{W}}_2^{\mathrm{o}}) \cdot \hat{\mathbf{W}}_1^{\mathrm{pred}\,\prime} \\ \hat{\mathbf{W}}_2^{\mathrm{o}} \cdot \hat{\mathbf{W}}_1^{\mathrm{pred}\,\prime} \end{bmatrix} = \boldsymbol{\xi}_{\mathrm{ST}}^{*\prime} \tag{28}$$

with

$$H_{\mathrm{ST}}^{\mathrm{o}} = [\hat{\mathbf{W}}_1^{\mathrm{o}}, \hat{\mathbf{W}}_2^{\mathrm{o}}, \hat{\mathbf{W}}_1^{\mathrm{o}} \times \hat{\mathbf{W}}_2^{\mathrm{o}}]^{\mathrm{T}} \tag{29}$$

The combined estimate of the attitude increment vector is then the value $\boldsymbol{\xi}^{*\prime}$ (recall equation (16a)) which minimizes the cost function

$$\begin{aligned} J(\boldsymbol{\xi}) = {} & \frac{1}{2}[\mathbf{Z}'_{\mathrm{ST}} - \boldsymbol{\xi}]^{\mathrm{T}} R_{\mathrm{ST}}^{-1} [\mathbf{Z}'_{\mathrm{ST}} - \boldsymbol{\xi}] \\ & + \frac{1}{2}[\hat{\mathbf{W}}'_S - A\hat{\mathbf{V}}_S]^{\mathrm{T}} R_S^{\#} [\hat{\mathbf{W}}'_S - A\hat{\mathbf{V}}_S] \end{aligned} \tag{30}$$

Data Fusion Within the Wahba Problem

When the Sun-sensor measurement is QUEST-like, one can calculate the combined estimate readily within the Wahba problem. Using equivalent directions, equation (24) becomes readily

$$J(A) = \frac{1}{2}\sum_{i=1}^{3} \frac{1}{(\sigma_i^2)^{\mathrm{eq}}} |\hat{\mathbf{W}}_i^{\mathrm{eq}\,\prime} - A\hat{\mathbf{V}}_i^{\mathrm{eq}}|^2 + \frac{1}{2}\frac{1}{\sigma_S^2}|\hat{\mathbf{W}}'_S - A\hat{\mathbf{V}}_S|^2 \tag{31}$$

and the optimal attitude estimate $A^{*\prime}$ can then be found using, for example, the QUEST algorithm [5]. The inverse covariance matrix for the combined estimate (equation (20)) becomes

$$\begin{aligned} P_{\mathrm{ST}}^{-1} &= R_{\mathrm{ST}}^{-1} + R_{S'}^{\#} \\ &= R_{\mathrm{ST}}^{-1} + \frac{1}{\sigma_S^2}(I_{3\times3} - \hat{\mathbf{W}}_S^{\mathrm{true}} \hat{\mathbf{W}}_S^{\mathrm{true T}}) \end{aligned} \tag{32}$$

Equivalently, one could define an effective attitude profile matrix B_{ST} for the star-tracker attitude estimate within the generalized Wahba problem [3, 6]

$$B_{\mathrm{ST}} = D_{\mathrm{ST}} C_{\mathrm{ST}}^{*\prime} \tag{33}$$

with D_{ST} given by equation (15). The total attitude profile matrix is[7]

$$B = B_{\mathrm{ST}} + B_S = B_{\mathrm{ST}} + \frac{1}{\sigma_S^2}\hat{\mathbf{W}}'_S \hat{\mathbf{V}}_S^{\mathrm{T}} \tag{34}$$

Reference [3] also contains formulae for generating the Davenport matrix K_{ST} directly from $C_{\mathrm{ST}}^{*\prime}$ and R_{ST}. In all cases the inverse attitude covariance matrix for the combined estimate is given by equation (31).

Clearly, equation (17) within general maximum-likelihood estimation and equation (34) within the Wahba problem provide the most efficient means for data fusion for this simple problem. The equivalent-vector approach, given by equation (24), or its equivalent in the Wahba problem if the Sun-sensor measurement is QUEST-like, provides some very visible insights into the problem, as we shall see later in this work.

[7]Note that the first term of equation (18) reduces readily to $\mathrm{tr}(D_{\mathrm{ST}}) - \mathrm{tr}(B_{\mathrm{ST}}^{\mathrm{T}} A)$.

Defective Directions

In contrast with the above methodologies, we examine the performance of two defective measurements. The Brozenec-Bender vector [10] is defective, because it discards portions of the measurement. The directions of the prescription of Bar-Itzhack and Harman [11] are defective, because that prescription makes impossible assumptions.

The Brozenec-Bender Vector

Brozenec and Bender in 1994 [10] proposed the approximate reduction of star-tracker data to an average observation vector and average reference vector

$$\hat{\bar{\mathbf{W}}}' \equiv \text{unit}\left(\sum_{k=1}^{N} \hat{\mathbf{W}}'_k\right) \qquad \text{and} \qquad \hat{\bar{\mathbf{V}}}' \equiv \text{unit}\left(\sum_{k=1}^{N} \hat{\mathbf{V}}'_k\right) \tag{35}$$

Here, unit(·) is a function which unitizes its argument. For a star tracker with a large number of observed star directions distributed uniformly over a field of view of 8 deg × 8 deg, it is easy to show [18] that to roughly one part in 1000, $\hat{\bar{\mathbf{W}}}$ conforms to the QUEST measurement model with variance parameter σ_{ST}^2/N, where σ_{ST}^2 is the single-star variance parameter. Thus

$$\hat{\bar{\mathbf{W}}} = A^{\text{true}}\hat{\bar{\mathbf{V}}} + \Delta\hat{\bar{\mathbf{W}}} \tag{36a}$$

$$\Delta\hat{\bar{\mathbf{W}}} \sim \mathcal{N}(\mathbf{0}, R_{\hat{\bar{\mathbf{W}}}}) \tag{36b}$$

$$R_{\hat{\bar{\mathbf{W}}}} = \frac{\sigma_{\text{ST}}^2}{N}(I_{3\times 3} - \hat{\bar{\mathbf{W}}}^{\text{true}}\hat{\bar{\mathbf{W}}}^{\text{true}\,\text{T}}) \tag{36c}$$

This approximation was shown to lead to an insignificant increase in the attitude error when the remaining sensors were of equal accuracy as the star tracker, but to a significant increase when this was not the case, for example in combining star-tracker attitudes with typical coarse Sun-sensor or horizon-scanner data [18]. For a spacecraft equipped with two star trackers with non-collinear boresights, the Brozenec-Bender approximation would work well [18]. The SCAD algorithm [19], inspired by reference [10], presented a method for retaining the Brozenec-Bender vectors as effective measurement and reference vectors but retaining most of the information lost in the data averaging. The Brozenec-Bender approximation cannot be said to be wrong in those cases where it leads to greater errors, only that its domain of applicability is limited.

The Prescription of Bar-Itzhack and Harman

Of a much different sort is the Bar-Itzhack-Harman (bih) prescription of reference [11], which proposed combining star-tracker and Sun-sensor data within the Wahba problem, in this case by representing the star-tracker attitude estimate by two effective direction measurements.

The algorithm of reference [11], unfortunately, is only a prescription presented without derivation, and the prescription turns out, as we shall see, to lead to an attitude estimate which can be *less accurate* when one includes the Sun-sensor data than when one simply discards it, contrary to the obvious purpose of data fusion. Nonetheless, the proposition of Bar-Itzhack and Harman of representing the star-tracker attitude by a small number of mutually-orthogonal unit-vector measurements consistent with the QUEST measurement model [3, 5, 6] was interesting, even inspired, and was, in fact, the motivation for the creation of both the equivalent directions [1] and the predicted directions [2].

The prescription of Bar-Itzhack and Harman begins with the star-tracker attitude estimate $C^{*\prime}_{\rm ST}$, the Sun-sensor measurement, $\hat{\mathbf{W}}'_S$, and the known Sun direction $\hat{\mathbf{V}}_S$. It then constructs two reference directions, $\hat{\mathbf{V}}_1^{\rm bih}$ and $\hat{\mathbf{V}}_2^{\rm bih}$, so that the triad $\{\hat{\mathbf{V}}_1^{\rm bih}, \hat{\mathbf{V}}_2^{\rm bih}, \hat{\mathbf{V}}_S\}$ is right-hand orthonormal. The effective star-tracker measurements are then given by

$$\hat{\mathbf{W}}_i^{\rm bih\prime} = C^{*\prime}_{\rm ST}\hat{\mathbf{V}}_i^{\rm bih}, \qquad i = 1, 2 \tag{37}$$

These values are then inserted into a Wahba-like cost function

$$J^{\rm bih}(A) = \frac{1}{2}\sum_{i=1}^{2}\frac{1}{\sigma^2_{\text{ST-bih}}}|\hat{\mathbf{W}}_i^{\rm bih\prime} - A\hat{\mathbf{V}}_i^{\rm bih}|^2 + \frac{1}{2}\frac{1}{\sigma^2_{\text{S-bih}}}|\hat{\mathbf{W}}'_S - A\hat{\mathbf{V}}_S|^2 \tag{38}$$

and the minimizing value $A^{*\prime\rm bih}$ is taken for the combined estimate of the spacecraft attitude based on the star-tracker attitude estimate and the Sun-sensor measurement.

We may infer from equations (37) and (38) that the Bar-Itzhack-Harman prescription makes the following implicit assumptions about the effective direction measurements, $\hat{\mathbf{W}}_1^{\rm bih}$ and $\hat{\mathbf{W}}_2^{\rm bih}$, which they propose to represent the star-tracker attitude:

- The effective measurements are two in number.
- Their directions may be chosen arbitrarily.
- The effective measurements are statistically independent.
- They are nay be used as inputs to the Wahba problem; hence, they are QUEST-like.

No set of effective measurements has these four properties. The predicted direction measurements have the first two properties but not the last two; the equivalent direction measurements have the last two properties but not the first two. Thus, the model of reference [11] is impossible, at least if we are to understand it as being meaningful within maximum-likelihood estimation. Furthermore, if we were to choose the body-axes so that the z-axis is along the measured Sun direction, then the star-tracker attitude covariance matrix computed from $\hat{\mathbf{W}}_1^{\rm bih}$ and $\hat{\mathbf{W}}_2^{\rm bih}$ as equivalent directions would be [5]

$$R^{\rm bih}_{\rm ST} = \sigma^2_{\text{ST-bih}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \tag{39}$$

which is clearly unphysical for a star tracker, which typically has a very narrow field of view and a variance about the boresight which is generally orders of magnitude larger than that about one of the focal-plane axes. In addition, there is no connection between the star-tracker boresight axes and the Sun direction. Within maximum-likelihood estimation, the prescription of reference [11] is not only impossible mathematically but unreasonable physically. It is simply a sequence of *ad hoc* operations which have no basis within estimation theory.

On the basis of the second implicit assumption above, namely, that the two effective directions can be chosen arbitrarily, which is crucial to the prescription of reference [11], we may discard the equivalent directions as a possible truth model for the covariance analysis. Thus, we shall carry out a covariance analysis of the Bar-Itzhack-Harman prescription based on the predicted direction measurements, which makes the neglect of correlations and the use of the Wahba problem and the neglect of correlation errors of implementation. There is no other alternative.

A second defect of the Bar-Itzhack-Harman prescription is that the coefficients in equation (38) do not correspond to the target mission, the Wilkinson Microwave Anisotropy Probe (WMAP) [20]. The parameter $\sigma_{\text{S-bih}}$ is more than an order of magnitude larger than the true value. For the star tracker, there is no single parameter characterizing the accuracy, since the errors about the boresight are generally more than an order of magnitude larger than those about a transverse axis. Were we to accept the values of $\sigma_{\text{ST-bih}}$ and $\sigma_{\text{S-bih}}$, we would predict estimation accuracies for the fused estimates which were an order of magnitude larger (in standard deviation) than is actually the case.

Nature of the Tests

We assume the QUEST measurement model [5] in our work as a model for the original measurements. Thus, all maximum-likelihood approaches to the fusion of star-tracker and Sun-sensor data in this work become applications of the Wahba problem [3, 4]. We, therefore, confine our studies to four methods: (1) the maximum-likelihood method, (2) The Brozenec-Bender approximation, (3) the prescription of Bar-Itzhack and Harman, and (4) discarding the Sun-sensor data entirely. As our test bed we will use the sensor suite of the Wilkinson Microwave Anisotropy Probe mission [20], which was also, supposedly, the target mission for the prescription of Bar-Itzhack and Harman. The covariance analysis of the prescription of reference [11] is nontrivial, because it is not a maximum-likelihood estimator given the original data, so that one cannot compute the covariance matrix from the cost function presumed by reference [11] by computing its Hessian matrix. In addition, as we have pointed out above, the measurements are mismodeled as well. Because the attitude estimate and attitude estimation-error covariance matrix can always be represented physically by equivalent directions, no matter what the nature of the original measurements, provided that the attitude is observable, we shall represent the star-tracker data by equivalent directions, because they offer physical insights. The Bar-Itzhack-Harman effective measurement vectors for the star tracker are treated as predicted directions, although they are consistent with no possible measurement model.[8]

The WMAP Mission

The WMAP attitude sensors consist of redundant star trackers (not used simultaneously) and two similarly redundant digital Sun sensors. The WMAP star tracker was a Lockheed-Martin AST-201 autonomous star tracker [21], capable of tracking as many as 50 stars simultaneously and typically tracking about 25 in the WMAP mission [20]. The typical attitude accuracy of this device is approximately $\tau_t = 2$ arcsec about axes transverse to the boresight and approximately $\tau_b = 20$ arcsec about the star-tracker boresight. We write τ_t^2 and τ_b^2 for the variances associated with the star-tracker attitude estimates, σ_{ST}^2 for the single-direction variance.[9] The Adcole Sun

[8]It might be noted that an earlier unpublished analysis of the Bar-Itzhack-Harman prescription on the basis of equivalent measurements led to much poorer performance than the analysis here in terms of predicted directions. If the effective measurements are modeled as equivalent directions, then the fusion of Sun-sensor data by that prescription leads to increases in the standard deviations of the attitude errors of an order of magnitude.
[9]The actual attitude estimation accuracies reported in reference [20] following several years of in-flight operation were 2.3 arc seconds and 21 arc seconds, respectively, but we shall use the approximate numbers given above in the text for greater simplicity in our calculations. Reference [21] reports slightly different values from inflight analysis based on a shorter period of mission experience.

sensor had an accuracy $\sigma_S = 20$ arcsec. The accuracy of the Lockheed-Martin star tracker transverse to the boresight corresponds to a single-star accuracy (in the QUEST measurement model) of $\sigma_{\rm ST} = 10$ arcsec. For simplicity, we will assume that the centroid of the observed star directions is always along the star-tracker boresight.[10]

Although we will use sensor parameters which are very close to those of the WMAP spacecraft, for convenience we will examine simpler mission geometries. We consider two numerical examples, in both of which we assume that the star-tracker boresight is always along the body x-axis ($\pm y$-axes for the real WMAP spacecraft). For Numerical Example I we assume that the Sun direction is along the z-axis. In Numerical Example II we assume that the Sun direction is along the $-x$*-axis*. The Sun direction of the actual WMAP spacecraft generally lies on a cone of half-angle 22.5 deg about the WMAP body z-axis, so our first numerical example is very much an average of the two scenarios. The advantage of the simplified examples is that the computations can be carried out with no more than a pocket calculator.

Numerical Example I

MLE Covariance Analysis Using the Equivalent Vector Representation

Suppose that we are given an attitude estimate $C_{\rm ST}^{*\prime}$ and attitude covariance matrix $R_{\rm ST}$ arising from the star-tracker data alone and a measurement of the Sun vector along the direction $\hat{\mathbf{b}}_3$ with QUEST measurement model variance σ_S^2. Using the equivalent-vector representation [1], the star-tracker attitude covariance matrix is given by

$$R_{\rm ST} = \tau_b^2 \hat{\mathbf{b}}_1 \hat{\mathbf{b}}_1^{\rm T} + \tau_1^2 (\hat{\mathbf{b}}_2 \hat{\mathbf{b}}_2^{\rm T} + \hat{\mathbf{b}}_3 \hat{\mathbf{b}}_3^{\rm T}) \tag{40}$$

Here, $\hat{\mathbf{b}}_1$ is the star-tracker boresight. In the notation of reference [3]

$$\tau_1^{\rm ST} = \tau_b = ({\rm GDOP})\tau_1 = 20 \text{ arcsec} = \sigma_S \tag{41a}$$

$$\tau_2^{\rm ST} = \tau_3^{\rm ST} = \tau_t = \sigma_{\rm ST}/\sqrt{N} = 2 \text{ arcsec} \tag{41b}$$

Using equations (22) we may read the equivalent observation vectors and equivalent variances directly from equations (40). The Wahba cost function (identical to that from treating the individual star observations explicitly), including the contribution of the Sun sensor is

$$J(A) = \frac{1}{2}\left(\frac{1}{\tau_t^2} - \frac{1}{2\tau_b^2}\right)|\hat{\mathbf{b}}_1 - A\hat{\mathbf{r}}_1|^2 + \frac{1}{4\tau_b^2}(|\hat{\mathbf{b}}_2 - A\hat{\mathbf{r}}_2|^2 + |\hat{\mathbf{b}}_3 - A\hat{\mathbf{r}}_3|^2)$$
$$+ \frac{1}{2\sigma_S^2}|\hat{\mathbf{b}}_S - A\hat{\mathbf{r}}_3|^2 \tag{42}$$

[10]The observed GDOP factor (defined as τ_b/τ_t) for the WMAP mission is $21/2.3 \approx 9.1$. Assuming that the star observations are distributed uniformly over the field of view (8.8 deg $\times$ 8.8 deg for the AST-201 star tracker), and the QUEST measurement model, leads to an anticipated GDOP factor of about 16. (For a star tracker with a small square field of view and the QUEST measurement model, one anticipates that ${\rm GDOP} = (6/a^2)^{1/2}$ with a the angle-equivalent (full) width of the field of view.) This discrepancy with our expectations for an ideal measurement model is likely due to either uncompensated errors in the star tracker or the unsuitability of the QUEST measurement model. Naively, one would expect uncompensated errors to increase GDOP, because attitude errors about the boresight are probably more sensitive to inadequacies in the error model. A narrowing of the effective field of view will also increase GDOP. Thus, the smaller value of GDOP goes against our naive anticipations, but our naive anticipations are not always a reliable guide.

where, for our example,

$$\hat{\mathbf{b}}_1 = \hat{\mathbf{1}} \equiv \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \hat{\mathbf{b}}_2 = \hat{\mathbf{2}} \equiv \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \hat{\mathbf{b}}_3 = \hat{\mathbf{b}}_S = \hat{\mathbf{3}} \equiv \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{43abc}$$

and

$$\hat{\mathbf{r}}_i = C_{\mathrm{ST}}^{*\prime\mathrm{T}} \hat{\mathbf{b}}_i, \qquad i = 1, 2, 3 \tag{44}$$

We write $\hat{\mathbf{b}}_S$ in the last term of equation (43c) rather than $\hat{\mathbf{b}}_3$ because they are statistically different quantities even though in our present example $\hat{\mathbf{b}}_S = \hat{\mathbf{3}}$. (In Numerical Example II, we will choose $\hat{\mathbf{b}}_S = -\hat{\mathbf{1}}$.)

From equation (42) we may compute readily the attitude covariance matrix, which must be simply

$$(P_{\tilde{\epsilon}\tilde{\epsilon}})^{\mathrm{MLE}} = ((R_{\mathrm{ST}})^{-1} + R_S^{\#})^{-1} \tag{45}$$

with

$$(R_{\mathrm{ST}})^{-1} = \mathrm{diag}[\tau_b^{-2}, \tau_t^{-2}, \tau_t^{-2}] \qquad \text{and} \qquad R_S^{\#} = \mathrm{diag}[\sigma_S^{-2}, \sigma_S^{-2}, 0] \tag{46ab}$$

Here, "diag" denotes an $n \times n$ diagonal matrix with diagonal elements given by the n arguments.

Evaluating these expressions leads to

$$(P_{\tilde{\epsilon}\tilde{\epsilon}})^{\mathrm{MLE}} = \mathrm{diag}\left[\left(\frac{1}{\tau_b^2} + \frac{1}{\sigma_S^2}\right)^{-1}, \left(\frac{1}{\tau_t^2} + \frac{1}{\sigma_S^2}\right)^{-1}, \tau_t^2\right] \tag{47abc}$$

Covariance Analysis Using the Star-Tracker Data Alone

In this case the attitude covariance matrix is simply

$$(P_{\tilde{\epsilon}\tilde{\epsilon}})^{\mathrm{ST}} = R_{\mathrm{ST}} = \mathrm{diag}[\tau_b^2, \tau_t^2, \tau_t^2] \tag{48}$$

Covariance Analysis of the Brozenec-Bender Approximation

One can also calculate readily the attitude covariance matrix for the Brozenec-Bender approximation for our example. The Brozenec-Bender cost function is simply

$$J^{BB}(A) = \frac{1}{2\tau_t^2} |\hat{\mathbf{b}}_1 - A\hat{\mathbf{r}}_1|^2 + \frac{1}{2\sigma_S^2} |\hat{\mathbf{b}}_3 - A\hat{\mathbf{r}}_3|^2 \tag{49}$$

for which the attitude covariance matrix is diagonal

$$(P_{\tilde{\epsilon}\tilde{\epsilon}})^{\mathrm{BB}} = \mathrm{diag}\left[\sigma_S^2, \left(\frac{1}{\tau_t^2} + \frac{1}{\sigma_S^2}\right)^{-1}, \tau_t^2\right] \tag{50abc}$$

Covariance Analysis of the Prescription of Bar-Itzhack and Harman

The analysis of the prescription of reference [11] will require considerable care, because it assumes parameter values different from those of the actual mission and also misapplies the Wahba cost function.

True Covariance Analysis of the Prescription of Bar-Itzhack and Harman

Accuracy evaluation for the prescription of reference [11] cannot be accomplished as easily as that of the MLE method, because that prescription does not correspond to the maximum-likelihood estimate. A true assessment of the attitude estimation accuracy of the prescription of reference [11] using the weights of reference [11] but a realistic model or the WMAP sensor errors is not difficult, however. To accomplish this, we begin by writing the (random) cost function of reference [11] as

$$J^{\text{bih}}(A) = \frac{1}{2}\sum_{k=1}^{3} a_k^{\text{bih}} |\hat{\mathbf{b}}_k^{\text{bih}} - A\hat{\mathbf{r}}_k|^2 \tag{51}$$

with

$$a_1^{\text{bih}} = a_2^{\text{bih}} = c/\sigma_{\text{ST-bih}}^2 \qquad \text{and} \qquad a_3^{\text{bih}} = c/\sigma_{\text{S-bih}}^2 \tag{52ab}$$

for some common c. We have added a superscript "bih" to emphasize that some of the measurements may be pseudo-measurements. The Bar-Itzhack-Harman attitude estimate $A^{\text{bih}*\prime}$ is the value of A which minimizes $J^{\text{bih}}(A)$. Reference [11] has assumed the values

$$\sigma_{\text{ST-bih}} = 25 \text{ arcsec} \qquad \text{and} \qquad \sigma_{\text{S-bih}} = 250 \text{ arcsec} \tag{53ab}$$

Thus,

$$a_1^{\text{bih}} : a_2^{\text{bih}} : a_3^{\text{bih}} = 1:1:0.01 \tag{54ab}$$

and the unit sum weights are approximately

$$a_1^{\text{bih}} = a_2^{\text{bih}} = a_{\text{ST}}^{\text{bih}} = 0.4975 \qquad \text{and} \qquad a_3^{\text{bih}} = a_S^{\text{bih}} = 0.005 \tag{55ab}$$

Writing

$$A = (I_{3\times 3} + [[\tilde{\boldsymbol{\epsilon}}_{\text{bih}}]])A^{\text{true}} \tag{56a}$$

$$\hat{\mathbf{b}}_k^{\text{true}} = A^{\text{true}}\hat{\mathbf{r}}_k \qquad \text{and} \qquad \Delta\hat{\mathbf{b}}_k^{\text{bih}} = \hat{\mathbf{b}}_k^{\text{bih}} - \hat{\mathbf{b}}_k^{\text{true}}, \qquad k = 1, 2, 3 \tag{56bc}$$

the estimator for the attitude-increment vector becomes

$$\tilde{\boldsymbol{\epsilon}}_{\text{bih}}^* = G^{-1}\sum_{k=1}^{3} a_k^{\text{bih}} [[\hat{\mathbf{b}}_k^{\text{true}}]]\Delta\hat{\mathbf{b}}_k^{\text{bih}} \tag{57}$$

with

$$\begin{aligned} G &= \sum_{k=1}^{3} a_k^{\text{bih}}(I - \hat{\mathbf{b}}_k^{\text{true}}\hat{\mathbf{b}}_k^{\text{trueT}}) \\ &= \text{diag}\left[a_2^{\text{bih}} + a_3^{\text{bih}}, a_1^{\text{bih}} + a_3^{\text{bih}}, a_1^{\text{bih}} + a_2^{\text{bih}}\right] \end{aligned} \tag{58}$$

Thus,

$$E\{\tilde{\boldsymbol{\epsilon}}_{\text{bih}}\} = \mathbf{0} \tag{59a}$$

$$E\{\tilde{\boldsymbol{\epsilon}}_{\text{bih}}\,\tilde{\boldsymbol{\epsilon}}_{\text{bih}}^{\text{T}}\} \equiv P_{\tilde{\epsilon}\tilde{\epsilon}}^{\text{bih}} = G^{-1}HG^{-1} \tag{59b}$$

and

$$H = \sum_{k=1}^{3}\sum_{l=1}^{3} a_k^{\text{bih}} a_l^{\text{bih}} [[\hat{\mathbf{b}}_k^{\text{true}}]] E\{\Delta\hat{\mathbf{b}}_k^{\text{bih}}\,\Delta\hat{\mathbf{b}}_l^{\text{bihT}}\} [[\hat{\mathbf{b}}_l^{\text{true}}]]^{\text{T}} \tag{60}$$

We must now construct a model for $\Delta\hat{\mathbf{b}}_k^{\text{bih}}$, $k = 1, 2$. We have in this example that $\hat{\mathbf{b}}_3^{\text{bih}} = \hat{\mathbf{b}}_S$ and given by the QUEST measurement model [3, 5, 6], so that

$$E\{\Delta\hat{\mathbf{b}}_3^{\text{bih}}\} = \mathbf{0} \qquad \text{and} \qquad E\{\Delta\hat{\mathbf{b}}_3^{\text{bih}}\,\Delta\hat{\mathbf{b}}_3^{\text{bihT}}\} = \sigma_S^2(I_{3\times3} - \hat{\mathbf{b}}_3^{\text{true}}\hat{\mathbf{b}}_3^{\text{trueT}}) \qquad (61\text{ab})$$

and $\hat{\mathbf{b}}_3^{\text{bih}}$ is statistically independent of $\hat{\mathbf{b}}_1^{\text{bih}}$ and $\hat{\mathbf{b}}_2^{\text{bih}}$. For $k = 1$ and $k = 2$, we note that

$$\hat{\mathbf{b}}_k^{\text{bih}} \equiv C_{\text{ST}}^{*}\hat{\mathbf{r}}_k \qquad (62)$$

a predicted direction. Thus,

$$\begin{aligned}\hat{\mathbf{b}}_k^{\text{bih}} &= e^{[[\boldsymbol{\epsilon}_{\text{ST}}^{*}]]}C_{\text{ST}}^{\text{true}}\hat{\mathbf{r}}_k = e^{[[\boldsymbol{\epsilon}_{\text{ST}}^{*}]]}\hat{\mathbf{b}}_k^{\text{true}} \\ &\equiv \hat{\mathbf{b}}_k^{\text{true}} + \Delta\hat{\mathbf{b}}_k^{\text{bih}}, \qquad k = 1, 2\end{aligned} \qquad (63)$$

with

$$\Delta\hat{\mathbf{b}}_k^{\text{bih}} = -[[\hat{\mathbf{b}}_k^{\text{true}}]]\boldsymbol{\epsilon}_{\text{ST}}^{*}, \qquad k = 1, 2 \qquad (64)$$

It follows that

$$E\{\Delta\hat{\mathbf{b}}_k^{\text{bih}}\Delta\hat{\mathbf{b}}_l^{\text{bihT}}\} = [[\hat{\mathbf{b}}_k^{\text{true}}]]R_{\text{ST}}[\hat{\mathbf{b}}_l^{\text{true}}]]^{\text{T}}, \qquad k, l = 1, 2 \qquad (65)$$

and

$$\begin{aligned}H &= \left(\sum_{k=1}^{2} a_k^{\text{bih}}[[\hat{\mathbf{b}}_k^{\text{true}}]]^2\right)R_{\text{ST}}\left(\sum_{l=1}^{2} a_l^{\text{bih}}[[\hat{\mathbf{b}}_l^{\text{true}}]]^2\right) + (a_3^{\text{bih}})^2\sigma_S^2(I_{3\times3} - \hat{\mathbf{b}}_3^{\text{true}}\hat{\mathbf{b}}_3^{\text{trueT}}) \\ &= (a_{\text{ST}}^{\text{bih}})^2\begin{bmatrix}\tau_b^2 & 0 & 0 \\ 0 & \tau_t^2 & 0 \\ 0 & 0 & 4\tau_t^2\end{bmatrix} + (a_S^{\text{bih}})^2\begin{bmatrix}\sigma_S^2 & 0 & 0 \\ 0 & \sigma_S^2 & 0 \\ 0 & 0 & 0\end{bmatrix}\end{aligned} \qquad (66\text{a})$$

$$G = \text{diag}\left[a_{\text{ST}}^{\text{bih}} + a_S^{\text{bih}},\, a_{\text{ST}}^{\text{bih}} + a_S^{\text{bih}},\, 2\,a_{\text{ST}}^{\text{bih}}\right] \qquad (66\text{b})$$

with R_{ST} given by equation (40). Substituting equations (41) and (55) leads to

$$P_{\tilde{\epsilon}\tilde{\epsilon}}^{\text{bih}} = \text{diag}\left[(P_{\tilde{\epsilon}\tilde{\epsilon}}^{\text{bih}})_{11},\, (P_{\tilde{\epsilon}\tilde{\epsilon}}^{\text{bih}})_{22},\, (P_{\tilde{\epsilon}\tilde{\epsilon}}^{\text{bih}})_{33}\right] \qquad (67)$$

with

$$(P_{\tilde{\epsilon}\tilde{\epsilon}}^{\text{bih}})_{11} = \frac{(a_{\text{ST}}^{\text{bih}})^2\tau_b^2 + (a_S^{bih})^2\sigma_S^2}{(a_{\text{ST}}^{\text{bih}} + a_S^{\text{bih}})^2} = \frac{\tau_b^2 + (0.01)^2\sigma_S^2}{(1.01)^2} \qquad (68\text{a})$$

$$(P_{\tilde{\epsilon}\tilde{\epsilon}}^{\text{bih}})_{22} = \frac{(a_{\text{ST}}^{\text{bih}})^2\tau_t^2 + (a_S^{\text{bih}})^2\sigma_S^2}{(a_{\text{ST}}^{\text{bih}} + a_S^{\text{bih}})^2} = \frac{\tau_t^2 + (0.01)^2\sigma_S^2}{(1.01)^2} \qquad (68\text{b})$$

$$(P_{\tilde{\epsilon}\tilde{\epsilon}}^{\text{bih}})_{33} = \frac{(a_{\text{ST}}^{\text{bih}})4\tau_t^2}{(2\,a_{\text{ST}}^{\text{bih}})^2} = \tau_t^2 \qquad (68\text{c})$$

Numerical Comparisons I

Star-Tracker Data Alone

Evaluating equation (48) for the WMAP parameters leads to

$$(P_{\tilde{\epsilon}\tilde{\epsilon}})^{\text{ST}} = \text{diag}\left[(20.00\text{ arcsec})^2, (2.00\text{ arcsec})^2, (2.00\text{ arcsec})^2\right] \qquad (69)$$

The MLE Result

Evaluating equations (47) for the WMAP parameters leads directly to

$$(P_{\tilde{\epsilon}\tilde{\epsilon}})^{\text{MLE}} = \text{diag}\left[(14.14\ \text{arcsec})^2, (1.99\ \text{arcsec})^2, (2.00\ \text{arcsec})^2\right] \qquad (70)$$

This is the best achievable accuracy, the Cramér-Rao lower bound.

The Brozenec-Bender Approximation

Substituting the WMAP parameters into equations (50) leads to

$$(P_{\tilde{\epsilon}\tilde{\epsilon}})^{\text{BB}} = \text{diag}\left[(20.00\ \text{arcsec})^2, (1.99\ \text{arcsec})^2, (2.00\ \text{arcsec})^2\right] \qquad (71)$$

which shows slight improvement over using the star-tracker data alone.

The Prescription of Bar-Itzhack and Harman

Substituting the correct WMAP statistical parameters into equations (67) and (68) but using the weights calculated from $\sigma_{\text{ST-bih}}$ and $\sigma_{\text{S-bih}}$ of reference [11] leads to

$$(P_{\tilde{\epsilon}\tilde{\epsilon}})^{\text{actual bih}} = \text{diag}\left[(19.8\ \text{arcsec})^2, (1.99\ \text{arcsec})^2, (2.00\ \text{arcsec})^2\right] \qquad (72)$$

Results for Numerical Comparisons I

For the MLE result, we see an improvement by a factor of $\sqrt{2}$ for the accuracy about the star-tracker boresight, because the star-tracker and Sun-sensor error levels are equal about that axis. The other data-fusion methods are all roughly equal in accuracy but less accurate than the MLE result for very different reasons. The Brozenec-Bender approximation discards the information about the star-tracker boresight, but it happens to be equal to that about the star-tracker boresight from the Sun sensor. The result is to make it the same as for the star-tracker attitude estimate alone. For the Bar-Itzhack-Harman prescription, the mismodeling inherent in the misapplication of the Wahba problem has done the same thing, but here it is the Sun-sensor data that has been discarded by underweighting.[11] Had it been the case that σ_S was substantially larger than τ_b, or had the angle between the Sun direction and the star-tracker boresight been different from 90 deg, the Brozenec-Bender approximation would have performed much more poorly. The same would have been true in the Bar-Itzhack-Harman prescription if a smaller weight for the star-tracker data in comparison with that for the Sun measurement had been chosen, in which case the fused estimate would have been less accurate than that for the star-tracker data alone. Since no criterion has been given by the authors of reference [11] for selecting the weights, we must conclude that the estimate accuracy for the Bar-Itzhack-Harman fused estimate is not poorer than that of the star-tracker estimate alone only by happenstance. At best, it would seem, the Bar-Itzhack-Harman prescription does not make the attitude accuracy worse than if data fusion had been abandoned entirely.

Numerical Comparisons II

Consider now the case where the Sun direction is antiparallel to the star-tracker boresight. As before, we assume that the centroid of the star observations is along

[11] Correct modeling would have led to equation (31) or (34).

the star-tracker boresight (still the WMAP *x*-axis to make the comparisons more transparent). With the Sun direction now $-\hat{\mathbf{b}}_1$, one obtains readily:

Star-Tracker Data Alone

$$(P_{\tilde{\epsilon}\tilde{\epsilon}})^{\mathrm{ST}} = \mathrm{diag}[\tau_b^2, \tau_t^2, \tau_t^2] = \mathrm{diag}[(20.00 \text{ arcsec})^2, (2.00 \text{ arcsec})^2, (2.00 \text{ arcsec})^2] \tag{73}$$

as before.

The MLE result

$$(P_{\tilde{\epsilon}\tilde{\epsilon}})^{\mathrm{MLE}} = \left[\tau_b^2, \left(\frac{1}{\tau_t^2} + \frac{1}{\sigma_S^2}\right)^{-1}, \left(\frac{1}{\tau_t^2} + \frac{1}{\sigma_S^2}\right)^{-1}\right] = \mathrm{diag}[(20.00 \text{ arcsec})^2, (1.99 \text{ arcsec})^2, (1.99 \text{ arcsec})^2] \tag{74}$$

The Brozenec-Bender Approximation

$$(P_{\tilde{\epsilon}\tilde{\epsilon}})^{\mathrm{BB}} = \mathrm{diag}\left[\infty, \left(\frac{1}{\tau_t^2} + \frac{1}{\sigma_S^2}\right)^{-1}, \left(\frac{1}{\tau_t^2} + \frac{1}{\sigma_S^2}\right)^{-1}\right] = \mathrm{diag}[(\infty, (1.99 \text{ arcsec})^2, (1.99 \text{ arcsec})^2] \tag{75}$$

For the Brozenec-Bender algorithm there is now no data to compensate for the loss of the "off-axis" star data, so that in the present example there is no information about $\hat{\mathbf{b}}_1'$.[12]

The Prescription of Bar-Itzhack and Harman

For this case the Sun direction is $-\hat{\mathbf{1}}$, and the direction of the effective star-tracker measurements are $\hat{\mathbf{2}}$ and $\hat{\mathbf{3}}$. This leads to

$$G = \mathrm{diag}[2\, a_{\mathrm{ST}}^{\mathrm{bih}}, a_{\mathrm{ST}}^{\mathrm{bih}} + a_S^{\mathrm{bih}}, a_{\mathrm{ST}}^{\mathrm{bih}} + a_S^{\mathrm{bih}}] \tag{76a}$$

$$H = (a_{\mathrm{ST}}^{\mathrm{bih}})^2 \begin{bmatrix} 4\tau_b^2 & 0 & 0 \\ 0 & \tau_t^2 & 0 \\ 0 & 0 & \tau_t^2 \end{bmatrix} + (a_S^{\mathrm{bih}})^2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sigma_S^2 & 0 \\ 0 & 0 & \sigma_S^2 \end{bmatrix} \tag{76b}$$

from which, recalling equation (59b),

$$(P_{\tilde{\epsilon}\tilde{\epsilon}})^{\text{actual bih}} = \mathrm{diag}[(20.00 \text{ arcsec})^2, (1.99 \text{ arcsec})^2, (1.99 \text{ arcsec})^2] \tag{77}$$

Results for Numerical Comparisons II

In this case, the MLE result, even though it achieves the greatest possible accuracy, the Cramér-Rao lower bound, can make only small improvements. The Brozenec-Bender approximation performs very poorly in this case, because there is now no attitude information at all about the star-tracker boresight. The Bar-Itzhack-Harman prescription performs identically to the MLE result in this case, because by happenstance $\sigma_{\text{ST-bih}}/\sigma_{\text{S-bih}} = \tau_t/\sigma_S$. In any event, the Bar-Itzhack-Harman prescription performs equivalently to the correct MLE result only when the MLE result can lead to no real improvement in accuracy over using just the star-tracker attitude estimate alone.

[12] Those authors would surely not have implemented their approximation in this case.

Discussion and Conclusions

The effects of various data-fusion methods have been examined for combining star-tracker and Sun-sensor measurements. The methods compared have been: (1) correct estimation of the attitude using all data in accordance with the methods of maximum-likelihood estimation; (2) the Brozenec-Bender approximation for compressing star-direction data; (3) the heuristic prescription of Bar-Itzhack and Harman; and (4) simply discarding the Sun data altogether. The star-tracker attitude estimate and attitude estimate-error covariance matrix have been represented using the equivalent-direction representation [1]. The equivalent-direction representation itself should not be viewed as a solution method but only as a convenient way of representing the star-tracker data for our analysis. The relative quality of methods (2) through (4) must be judged by the proximity of their results to the result for maximum-likelihood estimation, as determined by the true covariance matrix for each method.

The Brozenec-Bender approximation does not work well when other sensors do not dominate the attitude estimate about the star-tracker boresight, and so the poor results seen here were to be expected. The Brozenec-Bender approximation, however, was never intended for unrestricted application but only in the case where the remaining sensors provide much more information about the attitude about the star-tracker boresight, than the star tracker itself. It would, therefore, be wrong for us to discredit the Brozenec-Bender approximation for its very poor performance in Numerical Example II, or even for its performance in Numerical Example I.

For the WMAP mission parameters, the Bar-Itzhack-Harman prescription leads to no real improvement in the accuracy of their fused attitude estimate above that of the star-tracker attitude estimate alone. A significant failing of that prescription is that no algorithm is given for selecting the weights for the data and the misapplication of the Wahba cost function eliminates the possibility for any credibility in it. The choice $a_{\text{S-bih}}/a_{\text{ST-bih}} = \tau_t^2/\sigma_S^2$ would seem to be optimal for the Bar-Itzhack-Harman prescription for our particular example. This does not alter the fact that it is wrong, because it assumes impossible properties for its effective measurements and discards important attitude information arbitrarily. Although this did not occur in the present study, it is possible for the Bar-Itzhack-Harman prescription, by a less fortuitous choice of weights, to increase the error levels of the fused attitude estimate beyond those of the star-tracker attitude estimate alone, in contrary to the purpose of data fusion. At best, the Bar-Itzhack-Harman prescription does not make the attitude estimate less accurate than by doing nothing.

Our results show also the importance of realistic simulations. Reference [11] reports that in only a few cases were the attitude error levels not decreased by including Sun-sensor data. From this we infer that the simulations of reference [11] simply used the standard deviation $\sigma_{\text{ST-bih}}$ for its two “equivalent” star-tracker directions in its simulation of the star-tracker attitude, rather than simulating the individual star directions used by the star tracker in producing its estimate of the attitude and did little more than find the attitude solution which minimizes equation (38). Had the individual star directions been simulated, even with the assumed single-star variances of reference [11], the inappropriateness of a single standard deviation to characterize the star-tracker attitude error levels would have been immediately evident. Likewise, had the covariance matrix arising from the two “effective” star-tracker measurements of reference [11] been compared to that from the *N* original star directions, the unsuitability of the model would have been evident as well. A considerable amount of literature exists on this particular very simple data-fusion problem, none of which has been cited by reference [11], which cites no journal article after 1965 nor any reference after 1978. All of these neglected publications offer better solutions to the specific data-fusion problem of reference [11].

The most important lesson of the present study is that reliable attitude estimation algorithms (including data fusion) cannot be created solely on the basis of intuition and engineering judgment. They must be subjected to rigorous analytical scrutiny as well as computational examination. For too many decades now, insufficient attention has been paid to the study of attitude covariance. This is a neglect we cannot afford.

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