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Effective Direction Measurements for Spacecraft Attitude: II. Predicted Directions

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Your gracious self, embrace but my direction.

William Shakespeare (1564–1616)
The Winter's Tale, Act IV, scene iv

Abstract

The predicted directions are examined as effective measurements of spacecraft attitude. It is shown that two linearly independent predicted directions are a sufficient statistic for the attitude, and that these two effective measurements can be chosen to be arbitrary directions as long as they are not parallel or antiparallel. However, the correct implementation of the predicted directions in maximum-likelihood estimation of the attitude is complicated by the form of the covariance matrix for each and their mutual correlation. Thus, the predicted directions, which differ greatly from the equivalent direction measurements presented earlier, are not useful as a practical vehicle for attitude data fusion. Unlike the equivalent directions, the predicted directions are always physically meaningful. Unfortunately, they are also almost physically meaningless, as we shall see. Nonetheless, they are of obvious theoretical interest and worthy of a careful examination. The predicted directions can be shown to be "equivalent" measurements for the TRIAD algorithm.

Introduction

In a recent paper [1], we introduced the equivalent directions as a minimal set of effective measurements which reproduce a given spacecraft attitude estimate and attitude covariance matrix. These effective measurements had the important property that they conformed to the QUEST measurement model [2, 3], that is, were "QUEST-like." Hence, they can be used not only in general attitude estimation problems but also within the Wahba problem [2–5].[2] In the present work, we introduce a different set of effective measurements, the *predicted directions,* which

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have some similar and some different properties from those of the equivalent directions. Unlike the equivalent directions, of which there are three, there are only two predicted directions which are statistically non-redundant. Since we are familiar with the notion that only two vector measurements are needed to determine the attitude, the predicted directions have a definite intuitive appeal. The present work examines this new set of effective measurements in detail.

The common property of the equivalent directions and the predicted directions as effective measurements is that they each satisfy in practice

$$\hat{\mathbf{W}}_k^{\text{effective}\,\prime} \equiv A^{*\prime}\hat{\mathbf{V}}_k^{\text{effective}} \tag{1}$$

where $A^{*\prime}$ is the given attitude estimate,[3] a direction-cosine matrix, and the $\hat{\mathbf{V}}_k^{\text{effective}}$ are reference unit vectors. The index k runs from 1 to 3 for equivalent directions and from 1 to 2 for predicted directions, which remains to be demonstrated. For the predicted directions, as we shall show, the two reference directions can be chosen to be orthogonal. For the three equivalent directions, they *must* be orthogonal [1].

We make no assumptions about the statistical nature or dimensions of the original attitude measurements, other than that they were sufficient to produce an attitude estimate. The effective measurements which we construct in this work are simply a minimum set of direction measurements which reproduces a given attitude estimate and attitude covariance matrices in maximum-likelihood estimation.

The Development and Properties of Predicted Directions

Let $\mathbf{V}^{\text{pred}}$ be an arbitrary 3×1 column vector, and define $\mathbf{W}^{\text{pred}}$, a random *predicted* (column) vector, as[4]

$$\mathbf{W}^{\text{pred}} \equiv A^{*}\mathbf{V}^{\text{pred}} \tag{2}$$

Here, A^*, the attitude estimator, can be written as [8]

$$A^{*} = e^{[[\tilde{\boldsymbol{\epsilon}}^{*}]]}A^{\text{true}} \tag{3}$$

with A^{true} the true value of the attitude matrix (direction-cosine matrix), $\tilde{\boldsymbol{\epsilon}}$ the attitude error increment vector, and $\tilde{\boldsymbol{\epsilon}}^{*}$ its estimator. We expect with probability only infinitesimally less than unity that the attitude error estimate $\tilde{\boldsymbol{\epsilon}}^{*\prime}$ will be an infinitesimal vector.[5]

[2]Reference [5] presents a masterful overview of the many solutions of the Wahba problem. On its numerical results and their interpretation, see references [6] and [7].

[3]The attitude *estimator* A^* is a random proper orthogonal matrix whose realization, a non-random proper orthogonal matrix, is the attitude *estimate* $A^{*\prime}$. Random vectors (or matrices or scalars), bear no special marking and realizations of these various random variables bear a prime. Almost all relationships in this work are satisfied *mutatis mutandis* by both random variables and their realizations. Quantities marked "true" or functions only of true variables are clearly non-random. Reference vectors, denoted here by $\mathbf{V}$, are always non-random.

[4]Note that while the equivalent-direction random variables were developed from a specific attitude *estimate,* the predicted-direction random variables are developed from the attitude *estimator* itself. Thus, the statistical relationship of these two sets of effective measurements are qualitatively very different.

[5]Following earlier work, we write $\tilde{\boldsymbol{\epsilon}}$ for the error measured from A^{true} and $\boldsymbol{\epsilon}$ for the error measured from an attitude only close to the truth (such as $A^{*\prime}$). In general, we have preferred to write equations in terms of random variables rather than in terms of realizations, because sometimes equations for random variables also contain realizations of random variables (typically as "subtraction" points) making it easier to transform the random equation to the non-random equation than *vice versa.*

We can write also

$$\mathbf{W}^{\text{pred}} = A^{\text{true}}\mathbf{V}^{\text{pred}} + \Delta\mathbf{W}^{\text{pred}} \tag{4}$$

with the predicted error vector $\Delta\mathbf{W}^{\text{pred}}$ given by

$$\Delta\mathbf{W}^{\text{pred}} = (e^{[[\tilde{\boldsymbol{\epsilon}}^*]]} - I_{3\times3})A^{\text{true}}\mathbf{V}^{\text{pred}} \simeq [[\tilde{\boldsymbol{\epsilon}}^*]]\mathbf{W}^{\text{true}} = -[[\mathbf{W}^{\text{true}}]]\tilde{\boldsymbol{\epsilon}}^* \tag{5}$$

Here, $[[\mathbf{u}]]$ denotes the 3×3 antisymmetric matrix constructed according to[6]

$$[[\mathbf{u}]] \equiv \begin{bmatrix} 0 & u_3 & -u_2 \\ -u_3 & 0 & u_1 \\ u_2 & -u_1 & 0 \end{bmatrix} \tag{6}$$

Clearly

$$E\{\Delta\mathbf{W}^{\text{pred}}\} = \mathbf{0} \tag{7}$$

$$E\{\Delta\mathbf{W}^{\text{pred}}\Delta\mathbf{W}^{\text{pred}\,\text{T}}\} = [[\mathbf{W}^{\text{true}}]]P_{\tilde{\epsilon}\tilde{\epsilon}}[[\mathbf{W}^{\text{true}}]]^{\text{T}} \tag{8}$$

with

$$P_{\tilde{\epsilon}\tilde{\epsilon}} \equiv E\{\tilde{\boldsymbol{\epsilon}}^*\tilde{\boldsymbol{\epsilon}}^{*\text{T}}\} \tag{9}$$

the attitude covariance matrix. Here $E\{\cdot\}$ denotes the expectation.

Consider now a set of predicted vectors $\{\mathbf{W}_k^{\text{pred}} \mid k = 1, \ldots, n\}$ which are all related to the corresponding predicted reference vectors $\{\mathbf{V}_k^{\text{pred}} \mid k = 1, \ldots, n\}$ by the same (random) transformation matrix A^*. Geometrically, there can be no more than three linearly independent $\mathbf{V}_k^{\text{pred}}$ and, hence, no more than three linearly independent $\mathbf{W}_k^{\text{pred}}$. (Of course, we wish to avoid the case that the three predicted reference unit vectors are infinitesimally close to being linearly dependent.) We know also from purely geometrical considerations that there must be at least two predicted direction measurements in order that the attitude be observable. Without loss of generality, let us choose the $\{\mathbf{V}_k^{\text{pred}} \mid k = 1, 2, 3\}$ to be a right-hand orthonormal set of *predicted reference directions* $\{\hat{\mathbf{V}}_k^{\text{pred}} \mid k = 1, 2, 3\}$. Any other set of $\{\mathbf{V}_k^{\text{pred}} \mid k = 1, \ldots, n\}$ can be readily constructed from these and *vice versa.* It follows that the corresponding predicted direction measurements are also a right-hand orthonormal set $\{\hat{\mathbf{W}}_k^{\text{pred}} \mid k = 1, 2, 3\}$. Since now

$$\hat{\mathbf{W}}_3^{\text{pred}} = \hat{\mathbf{W}}_1^{\text{pred}} \times \hat{\mathbf{W}}_2^{\text{pred}} \tag{10}$$

exactly, it follows that there can be only two *non-redundant* predicted directions, that is, whose noise terms are each not a linear combination of those of the other(s).[7] Hence, A^* is characterized completely by $\hat{\mathbf{W}}_1^{\text{pred}}$ and $\hat{\mathbf{W}}_2^{\text{pred}}$, a restatement of the fact that a 3×3 proper orthogonal matrix is characterized completely by the result of its action on a set of two linearly independent 3×1 column vectors.

As a result of equations (2) through (9), the maximum-likelihood estimator of the attitude matrix given the two (random) predicted directions $\hat{\mathbf{W}}_1^{\text{pred}}$ and $\hat{\mathbf{W}}_2^{\text{pred}}$ as measurements must be the same A^* as was used to construct $\hat{\mathbf{W}}_1^{\text{pred}}$ and $\hat{\mathbf{W}}_2^{\text{pred}}$. We shall demonstrate this explicitly in the next section and learn much about the

[6]Some authors prefer to use $[\mathbf{u}\times] \equiv -[[\mathbf{u}]]$.

[7]The equivalent directions satisfy a relation similar to equation (10) but only for a single realization of the random equivalent-direction measurements.

implementation of these particular predicted directions in attitude estimation problems in the process.[8]

Maximum-Likelihood Estimation using Predicted Direction–General

We demonstrate here how one must treat predicted directions in maximum-likelihood estimation problems and illustrate the nature of the burden they impose.

Let us consider two linearly independent but otherwise arbitrary predicted reference directions $\hat{\mathbf{V}}_1^{\text{pred}}$ and $\hat{\mathbf{V}}_2^{\text{pred}}$ and the corresponding predicted directions $\hat{\mathbf{W}}_1^{\text{pred}}$ and $\hat{\mathbf{W}}_2^{\text{pred}}$. Then

$$\hat{\mathbf{W}}_k^{\text{pred}} = A^{\text{true}}\hat{\mathbf{V}}_k^{\text{pred}} + \Delta\hat{\mathbf{W}}_k^{\text{pred}}, \qquad k = 1, 2 \tag{11}$$

and

$$E\{\Delta\hat{\mathbf{W}}_k^{\text{pred}}\} = \mathbf{0}, \qquad k = 1, 2 \tag{12a}$$

$$E\{\Delta\hat{\mathbf{W}}_k^{\text{pred}}\Delta\hat{\mathbf{W}}_l^{\text{pred}\,\mathrm{T}}\} = [[\hat{\mathbf{W}}_k^{\text{true}}]]P_{\tilde{\boldsymbol{\epsilon}}\tilde{\boldsymbol{\epsilon}}}[[\hat{\mathbf{W}}_l^{\text{true}}]]^{\mathrm{T}}, \qquad k, l = 1, 2 \tag{12b}$$

The two predicted direction measurements are correlated!

The predicted directions are very different from the equivalent directions [1], of which there are usually three, and which satisfy, if the equivalent variances are positive,

$$\hat{\mathbf{W}}_k^{\text{eq}} = A^{\text{eq true}}\hat{\mathbf{V}}_k^{\text{eq}} + \Delta\hat{\mathbf{W}}_k^{\text{eq}} \equiv \hat{\mathbf{W}}_k^{\text{eq true}} + \Delta\hat{\mathbf{W}}_k^{\text{eq}}, \qquad k = 1, 2, 3 \tag{13}$$

and[9]

$$E\{\Delta\hat{\mathbf{W}}_k^{\text{eq}}\} = \mathbf{0}, \qquad k = 1, 2, 3 \tag{14a}$$

$$E\{\Delta\hat{\mathbf{W}}_k^{\text{eq}}\Delta\hat{\mathbf{W}}_l^{\text{eq}\,\mathrm{T}}\} = \delta_{kl}(\sigma_k^2)^{\text{eq}}[[\hat{\mathbf{W}}_k^{\text{eq true}}]]\,[[\hat{\mathbf{W}}_l^{\text{eq true}}]]^{\mathrm{T}}, \qquad k, l = 1, 2, 3 \tag{14b}$$

In writing equation (14b) we have noted the general relation

$$[[\mathbf{u}]][[\mathbf{v}]]^{\mathrm{T}} = \mathbf{u}^{\mathrm{T}}\mathbf{v}I_{3\times 3} - \mathbf{v}\mathbf{u}^{\mathrm{T}} \tag{15}$$

in order to emphasize the similarities and differences in equations (12) and (14). Equations (14) are just the QUEST measurement model [2, 3], which is the frequent model for the elemental vector measurements.[10] This was the whole point of the equivalent directions. In some special cases (when $1/(\sigma_k^2)^{\text{eq}} = 0$ for some k), there may be, effectively, only two equivalent directions.[11] Formally, there are always three. The equivalent directions are not correlated, and they have a simpler covariance matrix than that of the predicted direction measurements.[12] We stress that there is no connection between the equivalent reference directions, the "predicted" reference directions, and the reference directions corresponding to the original measurements.

[8]One could, of course, define a myriad of predicted directed measurements, but only two would be non-redundant statistically, and one could not use any of the rest as measurements in an estimation problem. In fact, if the attitude estimator errors were Gaussian, one could not write a joint attitude information matrix for more than two predicted directions, because of the unit correlations.

[9]Note that by explicit construction the $\Delta\hat{\mathbf{W}}_k^{\text{eq}}$, $k = 1, 2, 3$, are Gaussian, while this need not be true for the predicted directions $\Delta\hat{\mathbf{W}}_k^{\text{pred}}$, $k = 1, 2$, although frequently it will be at least approximately true.

[10]We use the expression *elemental measurements* to describe the original direction measurements from the attitude sensors, as opposed to *predicted direction measurements,* derived here from the attitude estimator.

[11]This will occur, for example, if A^* were calculated from two elemental direction measurements conforming to the QUEST measurement model [1].

[12]Simplicity is sometimes the result of artificiality, which we shall see to be the case here.

Note that we have not defined $A^{\text{pred true}} = A^{*\prime}$ and $\Delta\hat{\mathbf{W}}_i^{\text{pred}\prime} = \mathbf{0}$, $i = 1, 2$, for the realizations of interest in contrast to our procedure for the equivalent directions. Such a formulation is allowable, but would have proved inconvenient. This leads to the important difference that the truth model for the attitude is known *ipso facto* for the equivalent directions (where it is artificial) but not for the predicted directions.

Note also that for both the equivalent directions and the predicted directions, the covariance matrices are singular, an expression of the fact that $|\hat{\mathbf{W}}_k^{\text{eq}}| = 1$ and $|\hat{\mathbf{W}}_k^{\text{pred}}| = 1$.

Note that for Gaussian measurement noise and for $P_{\tilde{\boldsymbol{\epsilon}}\tilde{\boldsymbol{\epsilon}}}$ a multiple of the 3×3 identity matrix, the predicted directions will look very much like the equivalent directions. However, the similarity is imperfect. There are three statistically independent equivalent direction measurements, which are uncorrelated but only two statistically non-redundant predicted direction measurements, which are strongly correlated. In a way, the correlation makes up for having one less effective measurement, and *vice versa.*

The equivalent inverse variances, as we have seen in reference [1], can be negative, so that the equivalent directions are not always physically meaningful. The predicted directions, on the other hand, always have positive-semidefinite covariance matrices, and so are always physically meaningful. Unfortunately, they are also nearly meaningless, because the directions of the predicted directions can be chosen arbitrarily. Their meaning lies buried in the details of the covariance and cross-covariance matrices.

Reconstruction of the Attitude Estimator from Predicted Directions

Let $\hat{\mathbf{W}}_1^{\text{pred}}$ and $\hat{\mathbf{W}}_2^{\text{pred}}$ be two non-parallel random predicted direction measurements, which need not be mutually perpendicular. Then

$$\hat{\mathbf{W}}_k^{\text{pred}} = A^* \hat{\mathbf{V}}_k^{\text{pred}}, \qquad k = 1, 2 \tag{16}$$

From these sets of predicted direction measurements and corresponding direction measurements, we can construct orthonormal triads of random measurement directions and the corresponding reference directions according to the procedures of the TRIAD algorithm [2, 9–12]. Thus

$$\hat{\mathbf{s}}_1^{\text{pred}} \equiv \hat{\mathbf{W}}_1^{\text{pred}}, \qquad \hat{\mathbf{s}}_2^{\text{pred}} \equiv \frac{\hat{\mathbf{W}}_1^{\text{pred}} \times \hat{\mathbf{W}}_2^{\text{pred}}}{|\hat{\mathbf{W}}_1^{\text{pred}} \times \hat{\mathbf{W}}_2^{\text{pred}}|} \qquad \text{and} \qquad \hat{\mathbf{s}}_3^{\text{pred}} \equiv \hat{\mathbf{s}}_1^{\text{pred}} \times \hat{\mathbf{s}}_2^{\text{pred}} \tag{17abc}$$

and

$$\hat{\mathbf{r}}_1^{\text{pred}} \equiv \hat{\mathbf{V}}_1^{\text{pred}}, \qquad \hat{\mathbf{r}}_2^{\text{pred}} \equiv \frac{\hat{\mathbf{V}}_1^{\text{pred}} \times \hat{\mathbf{V}}_2^{\text{pred}}}{|\hat{\mathbf{V}}_1^{\text{pred}} \times \hat{\mathbf{V}}_2^{\text{pred}}|} \qquad \text{and} \qquad \hat{\mathbf{r}}_3^{\text{pred}} \equiv \hat{\mathbf{r}}_1^{\text{pred}} \times \hat{\mathbf{r}}_2^{\text{pred}} \tag{17def}$$

Defining, as usual

$$S^{\text{pred}} \equiv [\hat{\mathbf{s}}_1^{\text{pred}} \vdots \hat{\mathbf{s}}_2^{\text{pred}} \vdots \hat{\mathbf{s}}_3^{\text{pred}}] \qquad \text{and} \qquad R^{\text{pred}} \equiv [\hat{\mathbf{r}}_1^{\text{pred}} \vdots \hat{\mathbf{r}}_2^{\text{pred}} \vdots \hat{\mathbf{r}}_3^{\text{pred}}] \tag{18ab}$$

with each 3×3 matrix defined by its three column vectors, and each, by construction, proper orthogonal. It follows that

$$\hat{\mathbf{s}}_k^{\text{pred}} = A^* \hat{\mathbf{r}}_k^{\text{pred}}, \qquad k = 1, 2, 3 \tag{19a}$$

or, equivalently

$$S^{\text{pred}} = A^* R^{\text{pred}} \tag{19b}$$

and

$$A^* = S^{\text{pred}} R^{\text{predT}} = \sum_{k=1}^{3} \hat{\mathbf{s}}_k^{\text{pred}} \hat{\mathbf{r}}_k^{\text{predT}} \tag{20}$$

Since equations (16) are exact, so is equation (20).[13] Note that in this case we will obtain the same attitude estimator no matter which predicted direction we chose as $\hat{\mathbf{W}}_1^{\text{pred}}$.

Since the attitude estimator constructed from the random predicted direction measurements must be the same as the original attitude estimator constructed from the original random direction measurements, it follows that the attitude covariance matrix constructed from the predicted measurements must be the same as the original attitude covariance matrix. In fact, all attitude moments constructed from the random predicted directions must be the same as those constructed from the original random direction measurements.[14]

We see, therefore, that the random predicted direction measurements are *equivalent* for the random original direction measurements. The equivalent direction measurements were equivalent only for the original attitude estimate and attitude covariance matrix. The equivalence of the predicted direction measurements is thus much stronger than that for the equivalent direction measurements.

Entr'acte

It is wise to pause at this point and consider further the differences between the equivalent and the predicted directions. The equivalent-direction random measurements have by *fiat* a particular statistical character—they are QUEST-like. This imposes upon them a specific mode and measurement covariance matrix and also Gaussian statistics. Therefore, there is no connection between the statistics of the original random measurements or the attitude estimator that derives from them and the equivalent-direction random measurements and the attitude estimator that derives from the equivalent directions. The estimator from the original random measurements, whatever they may have been, and the estimator from the equivalent-direction random measurements agree only for a single realization of the original attitude estimator and the estimate from the realization of the equivalent random directions derived from that estimate. This makes the equivalent-direction *random* measurements highly artificial. There is no alternative if the equivalent-direction random variables are to conform to the Wahba problem, that wonderful *Tinker Toy* of attitude estimation.

The predicted-direction random measurements, on the other hand, are not constructed with the intent of conforming to one particular estimator, as was the case for the equivalent-direction random measurements. However, because of the homogeneity of the random measurement equations for the predicted directions, it turned out that the estimation method for proceeding from the predicted-direction random measurements to the attitude estimator was the TRIAD algorithm, an unexpected and marvelous result. There is a bijective map from the original attitude estimator, whatever its origin, to the estimator from the predicted-direction random measurement, and this map is satisfied for *every* realization of these two estimators,

[13]Note, in particular, that the condition for an unambiguous and exact TRIAD solution from equations (16), namely, $\hat{\mathbf{W}}_1 \cdot \hat{\mathbf{W}}_2 = \hat{\mathbf{V}}_1 \cdot \hat{\mathbf{V}}_2$, is always satisfied for predicted directions.

[14]Note that the $\hat{\mathbf{s}}_k^{\text{pred}}$, $k = 1, 2, 3$, are three predicted directions which are statistically redundant. However, we are using them only as constructs for the attitude estimate, not as measurements for maximum-likelihood estimation.

not just one. At the same time, the predicted-direction random measurements embody all of the complexity of the original random measurements; hence, there is no simple expression for the attitude covariance matrix for the predicted-direction random measurements, except, perhaps for a few very special cases. For the equivalent-direction random measurements, on the other hand, there is a simple formula for the attitude covariance matrix, namely, the QUEST formula of reference [2].

Our discussion above focused on the direction random variables. If we examine realizations of the equivalent-direction and predicted-direction random variables, then it is obvious that the realizations of the two predicted-direction random measurements, when inserted into the Wahba problem, will yield the correct corresponding attitude *estimate.* Likewise, any two of the $\hat{\mathbf{W}}_k^{\text{eq true}}$, $k = 1, 2, 3$, may be inserted into the TRIAD algorithm to obtain the correct corresponding attitude *estimate.* However, such interchanges are not possible for the direction *random* variables. The predicted-direction random-variables are not QUEST-like in general, and the equivalent-direction random-variables, in general, have the wrong statistics.

Computation of the Attitude Covariance Matrix from Predicted Directions

While the reconstruction of the attitude estimator from the random predicted direction measurements is simple and provided by the TRIAD algorithm, the calculation of the attitude covariance matrix from these predicted measurements is extremely complex. We know from experience [2, 10] that even calculating the attitude covariance matrix from two independent QUEST-like random direction measurements as inputs to the TRIAD algorithm is a formidable task, because the TRIAD algorithm does not provide the maximum-likelihood estimate given these two vector measurements.[15] Computing the attitude covariance for the TRIAD algorithm from two correlated random predicted direction measurements of arbitrary statistical nature would be overwhelming. If we limit ourselves to a quadratic expansion of the negative-log-likelihood function [13] of the attitude given the measurement noise, then we are, in effect, dealing only with a Gaussian approximation, since in such a Taylor expansion, only the measurement means and measurement covariance matrices will be important.[16] Despite the more general nature of the predicted direction measurements in theory, in practice one is stuck with a Gaussian approximation. This extraordinary burden does not occur for the equivalent directions for which we wish to compute only an approximate Gaussian QUEST-like replacement measurement model given a particular attitude estimate and attitude covariance matrix. In fact, were we given an arbitrary attitude estimator, we would have to take a given sample of it and compute the attitude covariance matrix before constructing the equivalent measurements.

To show the formidable nature of the calculation of an attitude covariance matrix, rather than $\hat{\mathbf{W}}_1^{\text{pred}}$ and $\hat{\mathbf{W}}_2^{\text{pred}}$, we may choose as the basis for our computation the

[15]As we have seen [12], the TRIAD attitude estimator can be made the maximum-likelihood estimator from a subset of these measurements.

[16]The obvious exception, of course, is when the covariance matrix is singular, in which case the third and fourth central moments, related to the skewness and kurtosis of the distribution, must be considered. Such treatment is needlessly complex. In those cases where the covariance matrices of the measurements are singular, it is best to remove the singularity (generally the result of a constraint) from the problem by a change of variables. Hence, one need never consider the case of a singular covariance matrix. For the Wahba problem, treated as a maximum-likelihood estimator, the measurement covariance matrices are singular, but because of the special nature of the measurement sensitivity matrices, these may be replaced by nonsingular matrices [3].

random vectors $\hat{\mathbf{s}}_1^{\text{pred}}$ and $\hat{\mathbf{s}}_2^{\text{pred}}$, which have the advantage of being mutually perpendicular. From equation (19a) the TRIAD predicted direction measurements satisfy

$$\hat{\mathbf{s}}_k^{\text{pred}} = A^{\text{true}}\hat{\mathbf{r}}_k^{\text{pred}} + \Delta\hat{\mathbf{s}}_k^{\text{pred}} = A^{\text{true}}\hat{\mathbf{r}}_k^{\text{pred}} - [[\hat{\mathbf{s}}_k^{\text{true}}]]\tilde{\boldsymbol{\epsilon}}, \qquad k = 1, 2, 3 \tag{21}$$

similar to the equations for the $\hat{\mathbf{W}}_k^{\text{pred}}$, $k = 1, 2$.

To sidestep the difficulties presented by the singularities of the covariance matrix matrices, we define two right-hand orthonormal triads,[17] $\{\hat{\mathbf{a}}_k, \hat{\mathbf{b}}_k, \hat{\mathbf{W}}_k^{\text{true}}\}$, $k = 1, 2$, and we define predicted-direction components

$$s_{ka}^{\text{pred}} \equiv \hat{\mathbf{b}}_k^{\text{T}}\hat{\mathbf{s}}_k^{\text{pred}}, \qquad \text{and} \qquad s_{kb}^{\text{pred}} \equiv -\hat{\mathbf{a}}_k^{\text{T}}\hat{\mathbf{s}}_k^{\text{pred}}, \qquad k = 1, 2 \tag{22ab}$$

Then, defining the attitude increment vector $\boldsymbol{\xi}$ by[18]

$$A = \exp([[\boldsymbol{\xi}]])A^{\text{true}} \tag{23}$$

with $\exp(\,\cdot\,)$ the matrix exponential function, it follows from equations (11) that[19]

$$s_{ka}^{\text{pred}} = \hat{\mathbf{a}}_k^{\text{T}}\boldsymbol{\xi}^{\text{true}} + \Delta s_{ka}^{\text{pred}} \qquad \text{and} \qquad s_{kb}^{\text{pred}} = \hat{\mathbf{b}}_k^{\text{T}}\boldsymbol{\xi}^{\text{true}} + \Delta s_{kb}^{\text{pred}}, \qquad k = 1, 2 \tag{24ab}$$

We need not consider the third component, the "constraint" measurements [12]

$$s_{kw}^{\text{pred}} \equiv \hat{\mathbf{s}}_k^{\text{true}\,\text{T}}\hat{\mathbf{s}}_k^{\text{pred}} = 1 + O|\tilde{\boldsymbol{\epsilon}}^*|^2, \qquad k = 1, 2 \tag{25}$$

because this component is insensitive to $\boldsymbol{\xi}$ at $\boldsymbol{\xi} = \mathbf{0}$. We note

$$E\{(\Delta s_{ka})^2\} = \hat{\mathbf{a}}_k^{\text{T}} P_{\tilde{\epsilon}\tilde{\epsilon}}\,\hat{\mathbf{a}}_k, \qquad E\{(\Delta s_{kb})^2\} = \hat{\mathbf{b}}_k^{\text{T}} P_{\tilde{\epsilon}\tilde{\epsilon}}\,\hat{\mathbf{b}}_k \tag{26ab}$$

$$E\{\Delta s_{ka}\,\Delta s_{kb}\} = \hat{\mathbf{a}}_k^{\text{T}} P_{\tilde{\epsilon}\tilde{\epsilon}}\,\hat{\mathbf{b}}_k, \qquad k = 1, 2 \tag{26c}$$

We can choose $\hat{\mathbf{a}}_k$ and $\hat{\mathbf{b}}_k$, $k = 1, 2$, so that

$$\hat{\mathbf{s}}_1^{\text{pred true}} = \hat{\mathbf{b}}_2 \equiv \hat{\mathbf{c}}_1 \qquad \hat{\mathbf{a}}_1 = \hat{\mathbf{s}}_2^{\text{pred true}} \equiv \hat{\mathbf{c}}_2 \tag{27ab}$$

$$\hat{\mathbf{b}}_1 = \hat{\mathbf{a}}_2 = \hat{\mathbf{S}}_1^{\text{pred true}} \times \hat{\mathbf{s}}_2^{\text{pred true}} = \hat{\mathbf{s}}_3^{\text{pred true}} \equiv \hat{\mathbf{c}}_3 \tag{27c}$$

and we may write[20]

$$y_1 \equiv s_{2b} = \hat{\mathbf{c}}_1^{\text{T}}\boldsymbol{\xi}^{\text{true}} + \Delta y_1, \qquad y_2 \equiv s_{1a} = \hat{\mathbf{c}}_2^{\text{T}}\boldsymbol{\xi}^{\text{true}} + \Delta y_2 \tag{28ab}$$

$$y_3 \equiv s_{1b} = \hat{\mathbf{c}}_3^{\text{T}}\boldsymbol{\xi}^{\text{true}} + \Delta y_3, \qquad y_4 \equiv s_{2a} = \hat{\mathbf{c}}_3^{\text{T}}\boldsymbol{\xi}^{\text{true}} + \Delta y_4 \tag{28cd}$$

As usual, from two direction measurements we have, equivalently, four scalar measurements.[21]

A special property of these four scalar effective measurements is that

$$y_3 = y_4 = \hat{\mathbf{c}}_3^{\text{T}}\boldsymbol{\xi}^{\text{true}} + \hat{\mathbf{c}}_3^{\text{T}}\tilde{\boldsymbol{\epsilon}}^* \tag{29}$$

[17]Note that $\hat{\mathbf{a}}_k$, $\hat{\mathbf{b}}_k$ and later $\hat{\mathbf{c}}_k$ are all non-random.

[18]The attitude increment vector $\boldsymbol{\xi}$ is similar to $\tilde{\boldsymbol{\epsilon}}$, the attitude increment error vector, which will have been the estimator of $\boldsymbol{\xi}$ obtained from an earlier process. The tilde reminds us that $\tilde{\boldsymbol{\epsilon}}$ is referred to true body axes.

[19]Naturally, $\boldsymbol{\xi}^{\text{true}} = \mathbf{0}$, but we retain its presence in equations (24) in order to be able to write the likelihood function for $\boldsymbol{\xi}$ later on.

[20]Clearly, from equations (28), we assume that the true attitude and the maximum-likelihood estimate of the attitude are separated by only an infinitesimal rotation. For $\text{tr}(P_{\tilde{\epsilon}\tilde{\epsilon}})$ sufficiently large, as in Scenario 2 of reference [5], this may not be a good approximation. We do not, however, anticipate such a contrived scenario in actual practice. In any event, the attitude covariance matrix for that scenario will have an attitude estimation accuracy about the worst-known axis of 10 deg, so that the magnitude of large angle corrections ($(10\text{ deg})^2 \approx 1.5$ deg) is hardly of consequence.

[21]The decomposition of $\hat{\mathbf{s}}_1^{\text{pred}}$ and $\hat{\mathbf{s}}_2^{\text{pred}}$ into scalar measurements is similar to that of an earlier study of the TRIAD algorithm [12]. The correspondences to the notation of reference [12] are $y_1 = z_4$, $y_2 = z_2$, $y_3 = z_1$ and $y_4 = z_5$.

so that y_3 and y_4 are identical measurements. Since we are dealing with a linear system (to $O(|\boldsymbol{\xi}|^2, |\tilde{\boldsymbol{\epsilon}}^*|^2)$, which we believe to be very very very small), we may replace y_3 and y_4 by $(y_3 \pm y_4)/2$, so that one of the new measurements vanishes identically, is not sensitive to the parameters to be estimated, and may be discarded. The other new measurement is identical to y_3. Argal, exit y_4 (stage left).[22]

We still have the problem that these three scalar measurements are correlated. To take account of this complication, we define the concatenated measurement

$$\mathbf{Y} \equiv \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -\hat{\mathbf{c}}_3 \cdot \hat{\mathbf{s}}_2^{\text{pred}} \\ -\hat{\mathbf{c}}_3 \cdot \hat{\mathbf{s}}_1^{\text{pred}} \\ \hat{\mathbf{c}}_2 \cdot \hat{\mathbf{s}}_1^{\text{pred}} \end{bmatrix} = H\boldsymbol{\xi}^{\text{true}} + \Delta\mathbf{Y} \tag{30}$$

with

$$\Delta\mathbf{Y} \equiv \begin{bmatrix} \Delta y_1 \\ \Delta y_2 \\ \Delta y_3 \end{bmatrix} = \begin{bmatrix} -\hat{\mathbf{c}}_3 \cdot \Delta\hat{\mathbf{s}}_2^{\text{pred}} \\ -\hat{\mathbf{c}}_3 \cdot \Delta\hat{\mathbf{s}}_1^{\text{pred}} \\ \hat{\mathbf{c}}_2 \cdot \Delta\hat{\mathbf{s}}_1^{\text{pred}} \end{bmatrix} \tag{31}$$

Here

$$H = [\hat{\mathbf{c}}_1, \hat{\mathbf{c}}_2, \hat{\mathbf{c}}_3]^{\text{T}} \qquad \text{and} \qquad \Delta\mathbf{Y} = H\tilde{\boldsymbol{\epsilon}}^* \tag{32ab}$$

The likelihood function [13] for $\boldsymbol{\xi}$ is

$$L(\boldsymbol{\xi}\,|\,\hat{\mathbf{W}}_1^{\text{pred}\prime}, \hat{\mathbf{W}}_2^{\text{pred}\prime}) = L(\boldsymbol{\xi}\,|\,\hat{\mathbf{s}}_1^{\text{pred}\prime}, \hat{\mathbf{s}}_2^{\text{pred}\prime}) = L(\boldsymbol{\xi}\,|\,\mathbf{Y}_1') = \mathrm{p}_{\mathbf{Y}}(\mathbf{Y}' - H\boldsymbol{\xi}) \tag{33}$$

As we have said, a Gaussian approximation is implicit in the calculation of the effective measurement $\mathbf{Y}$. Thus we may write effectively for the cost function $J_{\mathbf{Y}}(\boldsymbol{\xi})$

$$J_{\mathbf{Y}}(\boldsymbol{\xi}) = \frac{1}{2}(\mathbf{Y}' - H\boldsymbol{\xi})^{\text{T}} P_{\mathbf{YY}}^{-1}(\mathbf{Y}' - H\boldsymbol{\xi}) \tag{34}$$

with $P_{\mathbf{YY}}$ computed from equation (31).

Effective Direction Measurements as Random Variables

We examine here the statistical differences between the equivalent and the predicted direction from a somewhat different point of view. For both the equivalent directions and the predicted directions, we emphasized that while we could write formally

$$\hat{\mathbf{W}}_k^{\text{effective}} = A^{\text{effective true}}\hat{\mathbf{V}}_k^{\text{effective}} + \Delta\hat{\mathbf{W}}_k^{\text{effective}}, \qquad k = 1, \ldots, 2 \text{ or } 3 \tag{35}$$

only one set of realizations of the effective measurement noise was to be used. The realizations of the effective directions were created to reproduce *one* attitude estimate and *one* attitude covariance matrix.

To illustrate a situation where a generalization of the equivalent direction measurements beyond a single realization can lead to contradictions, consider equation (25) of reference [1], namely

$$A^{*\prime} = \sum_{i=1}^{3} \hat{\mathbf{W}}_i^{\text{eq true}} \hat{\mathbf{V}}_i^{\text{eq T}} \tag{36}$$

[22]On the use of "argal" for Latin "ergo," see reference [14].

It is tempting to assert

$$A^* = \sum_{i=1}^{3} \hat{\mathbf{W}}_i^{\text{eq}}\, \hat{\mathbf{V}}_i^{\text{eq}\,\text{T}} \tag{37}$$

as a formula for the attitude estimator. The $\hat{\mathbf{W}}_i^{\text{eq}}$, $i = 1, 2, 3$, are random vectors. This formula, however, cannot be correct. To see this, suppose that equation (37) were true and examine the equally appealing

$$\begin{aligned}\hat{\mathbf{W}}_i^{\text{pred}} &\equiv A^* \hat{\mathbf{V}}_i^{\text{pred}}, & i &= 1, 2\\ &= \sum_{k=1}^{3} \hat{\mathbf{W}}_k^{\text{eq}} \hat{\mathbf{V}}_k^{\text{eq}\,\text{T}} \hat{\mathbf{V}}_i^{\text{pred}}, & i &= 1, 2\end{aligned} \tag{38}$$

Let us suppose now that each of the $\hat{\mathbf{V}}_i^{\text{pred}}$, $i = 1, 2$, is equal to one of the $\hat{\mathbf{V}}_k^{\text{eq}}$, $k = 1, 2, 3$, say, $k = 1, 2$, a choice we can always make, because $\hat{\mathbf{V}}_1^{\text{pred}}$ and $\hat{\mathbf{V}}_2^{\text{pred}}$ are arbitrary. Then, for this case,

$$\hat{\mathbf{W}}_i^{\text{pred}} = \hat{\mathbf{W}}_i^{\text{eq}}, \qquad i = 1, 2 \tag{39}$$

which is impossible, because the $\hat{\mathbf{W}}_i^{\text{pred}}$, $i = 1, 2$, are correlated (except in the bizarre special case that the attitude covariance matrix vanishes), while the corresponding $\hat{\mathbf{W}}_i^{\text{eq}}$, $i = 1, 2$, are uncorrelated. Equation (37), therefore, cannot be correct. The reason for this is that the equivalent random measurement errors for the equivalent direction measurements have no connection in general to the original measurement errors except for the single realization leading to the given $A^{*\prime}$. What we can write instead of equation (37) is

$$A^{\text{eq}*} = \sum_{i=1}^{3} \hat{\mathbf{W}}_i^{\text{eq}} \hat{\mathbf{V}}_i^{\text{eq}\,\text{T}} \tag{40}$$

where $A^{\text{eq}*}$ is the attitude estimator for the (random) equivalent directions, which is not the same as A^*, because the randomness of $\hat{\mathbf{W}}_i^{\text{eq}}$, $i = 1, 2, 3$, does not arise from the measurement noise terms which are reflected in the randomness of A^*, but in the randomness of the newly conjured $\Delta\hat{\mathbf{W}}_i^{\text{eq}}$, $i = 1, 2, 3$, which are not a transformation of the original measurement noise. One realization of the original measurement noise contributed to one realization $A^{*\prime}$, which was used in the construction of $\hat{\mathbf{W}}_i^{\text{eq true}}$, $i = 1, 2, 3$, and that is all. A realization of the new estimated $A^{*\text{eq}}$ is not an estimate of the attitude for any real data. It is simply a device, if you wish, for computing the attitude covariance matrix for the equivalent directions. The first equality of equation (38) is correct, however, and we can write for the predicted attitude estimator

$$A^{\text{pred}*} = A^* = \hat{\mathbf{W}}_1^{\text{pred}} \hat{\mathbf{V}}_1^{\text{pred}\,\text{T}} + \hat{\mathbf{W}}_2^{\text{pred}} \hat{\mathbf{V}}_2^{\text{pred}\,\text{T}} + (\hat{\mathbf{W}}_1^{\text{pred}} \times \hat{\mathbf{W}}_2^{\text{pred}})(\hat{\mathbf{V}}_1^{\text{pred}} \times \hat{\mathbf{V}}_2^{\text{pred}})^{\text{T}} \tag{41}$$

if we know that the two predicted directions are orthogonal, that is, the two predicted reference vectors are orthogonal. In this case equation (41) is identical to equation (20) above. There is no relationship between the random portions of A^* and $A^{\text{eq}*}$. The errors in these quantities are independent random variables. Only the realization of the attitude error which leads to particular given realization $A^{*\prime}$ is related to the equivalent directions.[23] For the predicted directions we can identity

[23]See the discussion of equivalent directions as random variables in reference [1].

$A^{\text{pred}*}$ with A^*. Thus, the equivalent directions can reproduce only one value of the attitude estimate, while the predicted directions as random variables can reproduce the attitude estimator.

Predicted Directions in Data Fusion

The value of the predicted directions would seem to be in data fusion.[24] If one wishes to combine the estimate of the attitude from a star tracker (based, perhaps, on 100 elemental star-direction measurements) with other attitude data, then it is sufficient to create two predicted direction measurements and combine these with data from the other sensors.

The predicted directions, however, do not offer a very practical means for data fusion because of the considerable burden of generating these quantities, their covariance and cross-covariance matrices, and the treatment of singularities.[25] If one wished to use predicted directions in practical work, say, in combining a star-tracker attitude estimate with other attitude measurements, then one must find an attitude matrix A_o infinitesimally close to the true attitude and construct right-hand orthonormal triads $\{\hat{\mathbf{a}}_k^o, \hat{\mathbf{b}}_k^o, \hat{\mathbf{s}}_k^{o\,\text{pred}}\}$, $k = 1, 2$, where $\hat{\mathbf{s}}_k^{\text{pred}\,o} \equiv A_o \hat{\mathbf{r}}_k^{\text{pred}}$, and A_o is an *a priori* value of the attitude matrix believed to be close to A^{true}. Clearly A_o is nonrandom. We may write

$$A = A(\boldsymbol{\xi}) = e^{[[\boldsymbol{\xi}]]} A_o \tag{42}$$

In the most frequent data-fusion application, the most frequent set of measurements to be represented by predicted directions would be those of a star tracker. For these we will assume that the measurement errors of individual stars are independent and identically distributed and Gaussian as well, in which case we may write $\tilde{\boldsymbol{\epsilon}}^* \sim \mathcal{N}(\mathbf{0}, R_{\text{ST}})$.[26] Thus, we write the cost function for fusion of the star-tracker attitude estimate with other data as

$$\begin{aligned} J(\boldsymbol{\xi}) = {} & \frac{1}{2}[\mathbf{Z}'_{\text{ST}} - H^o_{\text{ST}}\boldsymbol{\xi}]^{\text{T}} R_{\text{ST}}^{-1} [\mathbf{Z}'_{\text{ST}} - H^o_{\text{ST}}\boldsymbol{\xi}] \\ & + \frac{1}{2}\sum_{i=1}^{n} [\mathbf{z}'_i - \mathbf{f}_i(A(\boldsymbol{\xi}))]^{\text{T}} R_i^{-1} [\mathbf{z}'_i - \mathbf{f}_i(A(\boldsymbol{\xi}))] \end{aligned} \tag{43}$$

and

$$\mathbf{Z}_{\text{ST}} = \begin{bmatrix} -(\hat{\mathbf{s}}_1^{\text{pred}\,o} \times \hat{\mathbf{s}}_2^{\text{pred}\,o}) \cdot \hat{\mathbf{s}}_2^{\text{pred}} \\ -(\hat{\mathbf{s}}_1^{\text{pred}\,o} \times \hat{\mathbf{s}}_2^{\text{pred}\,o}) \cdot \hat{\mathbf{s}}_1^{\text{pred}} \\ \hat{\mathbf{s}}_2^{\text{pred}\,o} \cdot \hat{\mathbf{s}}_1^{\text{pred}} \end{bmatrix} = H^o_{\text{ST}} \boldsymbol{\xi}^{\text{pred}}_{\text{ST}} \tag{44}$$

where the $\mathbf{z}_i$, $i = 1, \dots, n$, are the other (effective) measurements, all characterized by additive zero-mean Gaussian measurement noise with respective measurement covariance matrices R_i, $i = 1, \dots, n$. If one chooses $A_o = C^{*\prime}_{\text{ST}}$, then $\mathbf{Z}'_{\text{ST}} = H^o_{\text{ST}} \boldsymbol{\xi}^{\text{pred}\prime} = \mathbf{0}$. The general treatment of cost functions like that in equation (43) is examined in detail in reference [15].

[24]Obviously, there is little profit in simply calculating the attitude and the attitude covariance matrix a second time.
[25]The equivalent directions also present a considerable burden because of the need to compute the spectral decomposition of the attitude covariance matrix.
[26]Recall that we write R_{ST} for the attitude covariance matrix from the elemental star-tracker measurements only, and $P_{\tilde{\epsilon}\tilde{\epsilon}}$ for the attitude covariance matrix in general or for the entire set of measurements.

We note from equation (44) that the second and third components of $\mathbf{Z}_{\mathrm{ST}}$ are the active components of $\hat{\mathbf{s}}_1^{\mathrm{pred}}$. Thus, the construction of $\mathbf{Z}_{\mathrm{ST}}$ furnishes us with an effective direction measurement $\hat{\mathbf{s}}_1^{\mathrm{pred}}$ and an effective scalar measurement $(\hat{\mathbf{s}}_1^{\mathrm{pred\,o}} \times \hat{\mathbf{s}}_2^{\mathrm{pred\,o}}) \cdot \hat{\mathbf{s}}_2^{\mathrm{pred}}$. (We might use instead as this scalar measurement $(\hat{\mathbf{s}}_1^{\mathrm{pred}} \times \hat{\mathbf{s}}_2^{\mathrm{pred}}) \cdot \hat{\mathbf{s}}_2^{\mathrm{pred}}$, and not require the ancillary vectors $\hat{\mathbf{s}}_1^{\mathrm{pred\,o}}$ and $\hat{\mathbf{s}}_2^{\mathrm{pred\,o}}$, but the dependence on $\boldsymbol{\xi}$ and the effect of the correlations would be complex.)

Of course, $\mathbf{Z}_{\mathrm{ST}}$ will not be Gaussian in general, although it will have mean $\mathbf{0}$ and covariance matrix $H_{\mathrm{ST}}^{\mathrm{pred\,o}} R_{\mathrm{ST}} H_{\mathrm{ST}}^{\mathrm{oT}}$. When we make the Gaussian assumption embodied in equation (43), we are replacing C_{ST}^{*} by a Gaussian approximation not unlike $C_{\mathrm{ST}}^{\mathrm{eq}*}$. (They differ in that $C_{\mathrm{ST}}^{\mathrm{eq\,true}} = C_{\mathrm{ST}}^{*\prime}$ and not $C_{\mathrm{ST}}^{\mathrm{true}}$.)

Conclusions

We have seen that two predicted directions and their covariance matrices and cross-covariance matrix can always be found which, in maximum-likelihood estimation, will lead to any given attitude estimate and attitude covariance matrix. The reference directions (or, equally, the observation directions) for these predicted directions are arbitrary except that they must be linearly independent, that is, their included angle must be different from 0 or π. We have seen that the predicted directions are not QUEST-like, and, therefore, cannot be used in the Wahba problem. They also cannot be part of a neat, compact enhanced representation in the manner of the attitude profile matrix B, the Davenport matrix K, and the equivalent-vector representation. At the same time, they are more general then these representations, because they are not limited to "equivalent" Gaussian random measurements which can match only a single realization of the attitude estimator. They have already proved useful in one analysis [16]. An interesting property of the predicted directions is that they will produce the same attitude estimate and attitude covariance matrix not only within general maximum-likelihood estimation but also for the TRIAD algorithm.

One important deficiency of the predicted directions is that they are not transparent. They are, in fact, decidedly opaque. They may be chosen to have arbitrary directions (within the single condition above); hence, their directions need not correspond to any property of the spacecraft. Their covariance structure, certainly, provides no intuitive clues to the nature of the attitude estimate and the attitude covariance matrix. The equivalent direction measurements, on the other hand, if the given attitude estimate and attitude covariance matrix were produced by a star tracker with a narrow field of view, have an immediate connection to the star-tracker geometry. They correspond largely to the star-tracker boresight and transverse axes, and the equivalent variances will show the effect of geometrical distortion of precision (GDOP). They have the added advantage that they can be used in the Wahba problem.

As a practical vehicle for data fusion, the predicted directions, like the equivalent directions, are not very appealing. Their implementation is needlessly complex compared to the general method expressed by equations (66) through (70) of reference [17] (see also reference [15]).

A common feature of all practical data-fusion calculations is that we are forced for practical reasons to assume Gaussian statistics. In principle, the predicted directions, unlike the equivalent directions, preserve the statistical properties of the original attitude estimator. When actually we come to use the predicted directions, however, as in equation (43), we have no practical choice but to make a Gaussian approximation in order to be able to write a cost function in the form of equation (43). While we speak grandly of maximum-likelihood estimation or minimum-variance estimation, when it comes to actual computation, especially in mission support, practical considerations force us to do least-square estimation.

Perhaps, the greatest contribution of this and the preceding paper [1] is the demonstration that equations (1) and the attitude covariance matrix do not specify a unique effective measurement model, and that, in fact, there are at least two candidate models, the equivalent directions and the predicted directions, which reproduce a given attitude estimate and attitude covariance matrix within maximum-likelihood estimation yet have very different geometrical and statistical properties. The equivalent directions are statistically independent and may be used in the Wahba problem, they are generally three in number, their directions are fixed by the attitude covariance matrix, and they are not always physically meaningful. The predicted directions are two in number, are highly correlated, have arbitrary directions (as long as the two reference directions are linearly independent), are always physically meaningful (but mean nothing, because the two predicted directions can be chosen arbitrarily), and they cannot be used in the Wahba problem. These two very different sets of effective direction measurements exist even if the original data consisted only of QUEST-like measurements. Both sets of effective direction measurements are interesting. The predicted directions and the equivalent directions are compared in Table 1.

TABLE 1. Properties of Effective Direction Measurements

Property	Equivalent Directions	Predicted Directions
Number	3	2
Directions	fixed by covariance matrix	arbitrary
Statistically Independent	Yes	No
QUEST-like	Yes	No
Measurement Noise	artificial construct	from attitude error
Statistics	Gaussian by construction	from attitude error
No. of Random Noise Terms	6	3
Reproducibility of Attitude	one estimate only	estimator
Attitude Truth Model	known ipso facto	unknown
Meaningful as R. V.	No	Yes
Physically Meaningful	sometimes	always
Intuitive Content	strong[27]	weak
Physical Clarity	transparent[27]	opaque
Ease of Implementation	easy[28]	hard
Equivalent for	QUEST	TRIAD
Useful in Mission S/W	No	No
Useful in Analysis & Design	Yes[29]	Yes[29]
Enhanced Representation	Yes	No

[27]when physical.
[28]after the spectral decomposition of the attitude covariance matrix.
[29]see reference [17].

As a corollary to our statistical studies here and in reference [1], one can say that there exists no set of two effective attitude measurement vectors which can reproduce a given arbitrary attitude estimate and attitude covariance matrix, which can have arbitrary directions, and which may be used in the Wahba problem. An application of the predicted directions and the equivalent directions is presented in reference [16].

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