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Attitude Analysis in Flatland: The Plane Truth[1]

Malcolm D. Shuster[2]

Abstract

Many results in attitude analysis are still meaningful when the attitude is restricted to rotations about a single axis. Such a picture corresponds to attitude analysis in the Euclidean plane. The present report formalizes the representation of attitude in the plane and applies it to some well-known problems in a two-dimensional setting. In particular we carry out a covariance of a two-dimensional analogue of the algorithm OLAE, recently proposed by Mortari, Markley, and Junkins for optimal attitude determination.

Introduction

> *I call our world Flatland, not because we call it so, but to make its nature clearer to you, … who are privileged to live in Space.*
>
> –*A. Square in* Flatland [1]

The treatment of attitude, because of the nonlinearity and noncommutivity of the composition rule, is much more difficult to treat than position, for which components may be combined by simple addition. The complexity of the attitude composition rule leads to virtually all attitude problems being intrinsically three-dimensional or, in the case of the quaternion, four-dimensional. There is, however, a class of attitude problems which are much simpler, namely, single-axis problems, and the study of these will in many cases illuminate the more complex problems. The present report attempts to formalize such a treatment.

Attitude in Flatland

> *Having amused myself til a late hour with my favourite recreation of Geometry, I had retired to rest with an unsolved problem in my mind.*

Let us imagine that the world, Flatland, has only two dimensions and a constant isotropic Euclidean metric. Such a world was imagined by Edwin Abbott Abbott [1],

[1]Dedicated to John L. Junkins on the occasion of his sixtieth birthday.

[2]Director of Research, Acme Spacecraft Company, and acting manager, Planecraft Division, 13017 Wisteria Drive, Box 328, Germantown, MD 20874. email: mdshuster@comcast.net.

with the intent of satirizing the social and political foibles of his day as much as of clarifying the concepts related to the dimensionality of space. Our interest here is more limited than Abbott's. We develop the mathematical structure of Flatland somewhat further in order to better understand those aspects of attitude which do not depend on the dimensionality of space. The quotations which appear in this report are from reference [1]. Following Abbott we will refer to our three-dimensional world as *Space*.

In Flatland, vectors are, of course, two-dimensional

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{1}$$

The "dot" product takes the usual form

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 = \mathbf{u}^{\mathrm{T}}\mathbf{v} = \mathbf{v}^{\mathrm{T}}\mathsf{I}\,\mathbf{u} = \mathbf{v} \cdot \mathbf{u} \tag{2}$$

while the "cross product" is now a scalar

$$\mathbf{u} \times \mathbf{v} = u_1 v_2 - u_2 v_1 = \det[\mathbf{u} \vdots \mathbf{v}] = \mathbf{v}^{\mathrm{T}}\mathsf{J}\,\mathbf{u} = -\mathbf{v} \times \mathbf{u} \tag{3}$$

There is, therefore, no *vector* product, and as alternate names to scalar and vector products we might prefer symmetric and antisymmetric products. The lack of a meaningful vector product in two dimensions was ultimately the cause of many years of grief for Hamilton [2–4].

The *attitude matrix* in two dimensions is a 2×2 proper orthogonal matrix, A, which transforms column vectors in the usual way

$$\mathbf{W} = A\mathbf{V} \tag{4}$$

with

$$A^{\mathrm{T}}A = AA^{\mathrm{T}} = \mathsf{I} \tag{5}$$

$$\det A = +1 \tag{6}$$

where I denotes the 2×2 identity matrix[3]

$$\mathsf{I} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{7}$$

It is a simple matter to show that in two dimensions the attitude matrix may be represented as

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \tag{8}$$

and θ is the *angle of rotation*. If we define the matrix J according to

$$\mathsf{J} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{9}$$

which satisfies

$$\mathsf{J}^2 = -\mathsf{I} \tag{10}$$

[3]Note that we have used a bold *sans serif* font to designate I, the identity matrix, and J, the "square root" of $-\mathsf{I}$ (q.v. equation (9)), because of their special nature.

then *Euler's formula* becomes simply

$$A = \cos\theta\,\mathbf{I} + \sin\theta\,\mathbf{J} \tag{11}$$

which is much simpler than the three-dimensional form [5]. Note that $\mathbf{J}$ acting on a vector always generates a vector perpendicular to it. The matrices $\mathbf{I}$ and $\mathbf{J}$ in Flatland have an importance similar to that of the 3×3 identity matrix and the Levi-Civita symbol in Space [5]. They are, in fact, the representations of these objects in two dimensions.

If we define now for a scalar a

$$[\![a]\!] \equiv a\mathbf{J} \tag{12}$$

then trivially

$$[\![a]\!]\,[\![b]\!] = -ab\mathbf{I} \tag{13}$$

which again is much simpler than the three-dimensional variant, and Euler's formula becomes

$$A = \exp[\![\theta]\!] \tag{14}$$

as in Space.

Corresponding to the quaternion in Space, in Flatland we must be content with the *binion* (pronounced "BYE-nee-on").[4] The binion of rotation is defined as

$$\bar{q} \equiv \begin{bmatrix} \sin(\theta/2) \\ \cos(\theta/2) \end{bmatrix} \tag{15}$$

for which

$$\bar{q}^{\mathrm{T}}\bar{q} = 1 \tag{16}$$

We continue to use the notation $\bar{q}$ (rather than $\bar{b}$) in order to retain a greater resemblance to the equations in Space.

A binion of rotation differs from the general binion in that it has unit norm. The binion differs from a 2×1 array in that it possesses not only the additive operation (addition of components) but also the operation of binion multiplication. The binions of rotation (with domain S^1, the unit circle) form a group under binion multiplication. The general binions (with domain R^2, the plane) form a field under binion addition and multiplication. The general quaternions (domain R^4) form a

[4]The name *binion* is, in fact, a hybrid word, like *quaternion*. The word *quaternion* is built on the Latin distributive numerical adjective *quaterni*, which means literally "by fours" (as in: marching by fours). Thus, a quaternion is an entity consisting of a grouping of things by fours, an apt descriptor. The corresponding Latin distributive adjective for two things is *bini*. Thus, the planar analogue of the quaternion is called a *binion* (pronounced "BYE-nee-on" (q.v. *binary*)). In reference [6] below, we consciously rejected *binion* in factor of the more vulgar *biernion* because of the rhyme. In Classical Persian, by the way, a set of four things is a *rubai*, a word of obvious Arabic origin, and a collection of *rubai*, in the sense of four-line poems (quatrains (sic)), is a *rubaiyyat*, the most famous being that of the celebrated poet-astronomer Omar Khayyam (1050?–1123).

It might be mentioned that *quaternion* is also a barbarism of sorts, since it combines a Latin adjective with a Greek noun suffix. A more consistent name, totally Latin in character, would have been *quaternium*, which has the misfortune of sounding like the name of a metal, lost somewhere, one might suppose, among the rare earth elements. This may have been Hamilton's reason for the hybrid nomenclature, since he was also a remarkable linguist, proficient by age 16 not only in many modern European languages, Latin, Greek, and Hebrew, but also in Old Irish, Arabic, Sanscrit, and Classical Persian. Ancient Greek did not possess numerical distributive adjectives. If one had been at Hamilton's disposal, perhaps we would be speaking now of something like *tetrakons* (built here on the Greek numerical adverb τετράκις) or *tettarons* (built here on the Greek cardinal number τέτταρες, τέτταρα) instead of *quaternions*.

non-commutative field (division ring, skew field) under quaternion addition and multiplication. A binion created from a 2×1 array $\mathbf{a}$ will sometimes be denoted by $\bar{\mathbf{a}}$.

In terms of the binion Euler's formula becomes

$$\begin{aligned} A(\bar{q}) &= (q_2^2 - q_1^2)\mathbf{I} + 2q_2 q_1 \mathbf{J} \\ &= (q_2^2 - q_1^2)\mathbf{I} + 2q_2 [\![q_1]\!] \\ &= (q_2 \mathbf{I} + q_1 \mathbf{J})^2 \\ &\equiv \{\bar{q}\}^2 \end{aligned} \tag{17}$$

The binion may be extracted from the attitude matrix in a manner similar to the method for extracting the quaternion from the attitude matrix in Space through

$$q_2 = \frac{1}{2}\sqrt{2 + \mathrm{tr}A}, \quad q_1 = \frac{1}{4q_2}(A_{12} - A_{21}) \quad \text{if} \quad q_2 \neq 0 \tag{18ab}$$

or

$$q_1 = \frac{1}{2}\sqrt{2 - \mathrm{tr}A}, \quad q_2 = \frac{1}{4q_1}(A_{12} - A_{21}) \quad \text{if} \quad q_1 \neq 0 \tag{18cd}$$

Here

$$\mathrm{tr}A \equiv A_{11} + A_{22} \tag{19}$$

Binion composition follows directly from the trigonometric formula and reads

$$\bar{q}'' = \bar{q}' \otimes \bar{q} = \{\bar{q}'\}\bar{q} = \{\bar{q}\}\bar{q}' \tag{20}$$

where, as in equation (17)

$$\{\bar{q}\} \equiv \begin{bmatrix} q_2 & q_1 \\ -q_1 & q_2 \end{bmatrix} = q_2 \mathbf{I} + q_1 \mathbf{J} \tag{21}$$

Note that binion composition is commutative, as is the multiplication of attitude matrices in two dimensions.

The binion propagation matrix $\{\bar{q}\}$ has several interesting and useful properties, which are listed in the appendix.

The *Gibbs scalar* or *Rodrigues scalar* is given by

$$g = q_1/q_2 = \tan(\theta/2) \tag{22}$$

Thus

$$\bar{q} = \frac{1}{\sqrt{1 + g^2}} \begin{bmatrix} g \\ 1 \end{bmatrix} \tag{23}$$

and Cayley's formula takes the familiar form

$$A = \frac{\mathbf{I} + [\![g]\!]}{\mathbf{I} - [\![g]\!]} \tag{24}$$

The composition of Gibbs scalars is given by

$$g'' = g' \circ g = \frac{g' + g}{1 - g'g} \tag{25}$$

in complete analogy to the formula for the Gibbs vector in Space.

The *Cayley-Klein parameters* are

$$\alpha \equiv q_2 + i q_1 = e^{i\theta/2} \quad \text{and} \quad \beta \equiv q_2 - i q_1 = e^{-i\theta/2} = \alpha^c \tag{26ab}$$

and the superscript c denotes complex conjugation. These obviously satisfy

$$\alpha\beta = 1 \tag{27}$$

It follows that

$$A = \frac{1}{2}(\alpha^2 + \beta^2)\mathbf{I} + \frac{1}{2i}(\alpha^2 - \beta^2)\mathbf{J} \tag{28}$$

Attitude Kinematics in Flatland

Restraining my impatience—for I was now under a strong temptation to rush blindly at my visitor and precipitate him into Space …

The kinematic equation for the attitude matrix is given as usual by

$$\frac{\mathrm{d}}{\mathrm{d}t} A(t) = [\![\omega(t)]\!] A(t) \tag{29}$$

which, in fact, defines $\omega(t)$. If we define the binion analogue

$$\bar{\omega} \equiv \begin{bmatrix} \omega \\ 0 \end{bmatrix} \tag{30}$$

then the kinematic equation for the binion is simply

$$\frac{\mathrm{d}}{\mathrm{d}t} \bar{q}(t) = \frac{1}{2} \bar{\omega}(t) \otimes \bar{q}(t) = \frac{1}{2} \Omega(\omega(t)) \bar{q}(t) \tag{31}$$

where

$$\Omega(\omega) \equiv \omega \mathbf{J} \tag{32}$$

Likewise, we can partition $\{\bar{q}\}$ defined by equation (21) in terms of column matrices as

$$\{\bar{q}\} \equiv [\Xi(\bar{q}) \;\; \bar{q}] \tag{33}$$

which leads to

$$\frac{\mathrm{d}}{\mathrm{d}t} \bar{q}(t) = \frac{1}{2} \Xi(\bar{q}(t)) \omega \tag{34a}$$

and

$$\Xi(\bar{q}) = \begin{bmatrix} q_2 \\ -q_1 \end{bmatrix} = \mathbf{J} \bar{q} \tag{34b}$$

The kinematic equation for the Gibbs scalar becomes finally

$$\frac{\mathrm{d}}{\mathrm{d}t} g(t) = \frac{1}{2}[1 + g^2(t)]\, \omega(t) \tag{35}$$

while that for the angle of rotation is just

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega \tag{36}$$

Euler's equation for rigid-body dynamics is simply

$$I\,\frac{\mathrm{d}\omega}{\mathrm{d}t} = N \tag{37}$$

where N, the *torque*, is a scalar and I, the *moment of inertia*, another scalar, is given by

$$I = \int r^2\,\mathrm{d}m \tag{38}$$

Attitude Errors in Flatland

If Fog were non-existent, all lines would appear equally and indistinguishably clear.

The representation of attitude errors in Flatland follows that in Space, with obvious simplifications. The error in the attitude matrix, since it has only a single degree of freedom, can be written as

$$A^* = (\delta A)\,A^{\mathrm{true}} \tag{39}$$

with A^* a random variable, usually an attitude estimate, and δA is the (multiplicative) attitude error

$$\delta A = \exp\{[\![\,\Delta\xi\,]\!]\} \approx \mathbf{I} + [\![\,\Delta\xi\,]\!] \tag{40}$$

with $\Delta\xi$, th attitude error angle, generally an infinitesimal quantity assumed to have zero mean. The attitude variance is defined to be

$$P_{\xi\xi} = E\{(\Delta\xi)^2\} \tag{41}$$

where $E\{\cdot\}$ denotes the expectation.

The modeling of vector measurement errors follow a similar pattern. We write

$$\hat{\mathbf{W}} = e^{[\![\varepsilon]\!]}A\hat{\mathbf{V}} \tag{42}$$

where ε is a zero-mean random variable with variance σ_W^2. In general, we assume the reference vector $\hat{\mathbf{V}}$ to be error free. (This may not be true, in fact, since for measurements of the geomagnetic field direction for example, it is the reference vector error which dominates the measurement error, but it simplifies the presentation of algorithm development.) In linear approximation this may be written as

$$\hat{\mathbf{W}} = A\hat{\mathbf{V}} + \Delta\hat{\mathbf{W}} \tag{43}$$

with

$$\Delta\hat{\mathbf{W}} = \varepsilon\,\mathbf{J}A\hat{\mathbf{V}} = [\![\,A\hat{\mathbf{V}}\,]\!]\varepsilon \tag{44}$$

and we have defined $[\![\,\mathbf{v}\,]\!]$ with *vector* argument to be

$$[\![\,\mathbf{v}\,]\!] \equiv \mathbf{J}\,\mathbf{v} = \begin{bmatrix} v_2 \\ -v_1 \end{bmatrix} \tag{45}$$

Thus,

$$[\![\mathbf{u}]\!]^{\mathrm{T}}\mathbf{v} = -\mathbf{u} \times \mathbf{v} = -\mathbf{u}^{\mathrm{T}}[\![\mathbf{v}]\!] \tag{46a}$$

$$[\![\mathbf{v}]\!]^{\mathrm{T}}\mathbf{v} = \mathbf{0} \tag{46b}$$

$$[\![\mathbf{u}]\!]^{\mathrm{T}}[\![\mathbf{v}]\!] = \mathbf{u} \cdot \mathbf{v} \tag{46c}$$

$$[\![\mathbf{u}]\!]\,[\![\mathbf{v}]\!]^{\mathrm{T}} = (\mathbf{u} \cdot \mathbf{v})\mathbf{I} - \mathbf{v}\mathbf{u}^{\mathrm{T}} \tag{46d}$$

$$[\![a]\!]\mathbf{v} = a[\![\mathbf{v}]\!] \tag{46e}$$

It is easy to show that

$$R_{\hat{\mathbf{W}}} \equiv E\{\Delta\hat{\mathbf{W}}\Delta\hat{\mathbf{W}}^{\mathrm{T}}\} = \sigma_W^2(\mathbf{I} - (A\hat{\mathbf{V}})(A\hat{\mathbf{V}})^{\mathrm{T}}) \tag{47}$$

Note that $[\![\mathbf{v}]\!]$ in Flatland is not a square matrix as it is in Space.

Batch Attitude Determination in Flatland

I answer that though we cannot see angles, we can infer them, and this with great precision.

We can now examine some well-known algorithms in their Flatland setting. These are the DYAD algorithm, the two-dimensional analogue of the TRIAD algorithm [7, 8], and the BEST algorithm, the two-dimensional analogue of the QUEST algorithm [8]. The development of these algorithms in two dimensions is very similar to that of their forbears in Space. As can be expected, the results are much simpler in the smaller dimension.

The DYAD Algorithm

For the DYAD algorithm[5] we seek an attitude matrix which satisfies

$$\mathbf{W} = A\mathbf{V} \tag{48}$$

where $\mathbf{V}$ and $\mathbf{W}$ are arbitrary vectors. In a space of n dimensions, $n - 1$ linearly independent vector measurements are required to uniquely determine the attitude matrix [9]. In two dimensions, therefore, a single measurement suffices. (In one dimension, zero measurements are sufficient.)

To construct the attitude matrix we first construct orthonormal dyads of reference and observation vectors as

$$\hat{\mathbf{r}}_1 = \frac{\hat{\mathbf{V}}}{|\hat{\mathbf{V}}|} \quad \text{and} \quad \hat{\mathbf{r}}_2 = \mathbf{J}\hat{\mathbf{r}}_1 \tag{49ab}$$

and

$$\hat{\mathbf{s}}_1 = \frac{\hat{\mathbf{W}}}{|\hat{\mathbf{W}}|} \quad \text{and} \quad \hat{\mathbf{s}}_2 = \mathbf{J}\hat{\mathbf{s}}_1 \tag{49cd}$$

From

$$\mathbf{J}^3 = -\mathbf{J} \tag{50}$$

it follows that

$$\mathbf{J}A\mathbf{J}^{\mathrm{T}} = A \quad \text{and} \quad A\mathbf{J}A^{\mathrm{T}} = \mathbf{J} \tag{51ab}$$

[5]F. Landis Markley has suggested that the DYAD algorithm be renamed the BAD algorithm in contrast with BEST.

Hence,

$$\hat{\mathbf{s}}_k = A\hat{\mathbf{r}}_k, \quad i = 1, 2 \tag{52}$$

Defining now orthogonal matrices (labeled by their columns)

$$M_R = [\hat{\mathbf{r}}_1 \quad \hat{\mathbf{r}}_2] \quad \text{and} \quad M_S = [\hat{\mathbf{s}}_1 \quad \hat{\mathbf{s}}_2] \tag{53}$$

it follows that

$$M_S = A M_R \tag{54}$$

whence

$$A = M_S M_R^{\mathrm{T}} \tag{55}$$

The development of the DYAD attitude variance follows almost identical steps as in the calculation of the TRIAD attitude covariance in Space [8] with the result

$$P^{\mathrm{DYAD}} = \sigma_W^2 \tag{56}$$

The BEST Algorithm

The BEST (Binion ESTimator) algorithm in Flatland is only slightly less complicated than the QUEST algorithm in Space. As usual, we seek an attitude matrix which minimizes the Wahba cost function [8, 10, 11]

$$J(A) = \frac{1}{2}\sum_{k=1}^{n} a_k |\hat{\mathbf{W}}_k - A\hat{\mathbf{V}}_k|^2 \tag{57}$$

where the a_k, $k = 1, \ldots, n$, $n \geq 2$, are a set of positive weights, whose sum, we will assume, is unity. As in the Space we define a gain function, $g(A)$, such that

$$g(A) = 1 - J(A) = \mathrm{tr}(B^{\mathrm{T}}A) \tag{58}$$

which is a maximum when $J(A)$ is a minimum, and, as before, the attitude profile matrix is given by

$$B = \sum_{k=1}^{n} a_k \hat{\mathbf{W}}_k \hat{\mathbf{V}}_k^{\mathrm{T}} \tag{59}$$

Substituting equation (17) in equation (58) leads straightforwardly to

$$g(\bar{q}) = (q_2^2 - q_1^2)s + 2q_1q_2z \tag{60}$$

where

$$s \equiv \mathrm{tr}(B^{\mathrm{T}}) = \mathrm{tr}B = B_{11} + B_{22} \tag{61a}$$

$$z \equiv \mathrm{tr}(\mathbf{J}B^{\mathrm{T}}) = -\mathrm{tr}(\mathbf{J}B) = B_{12} - B_{21} \tag{61b}$$

Thus

$$g(\bar{q}) = \bar{q}^{\mathrm{T}}K\bar{q} \tag{62}$$

with

$$K = \begin{bmatrix} -s & z \\ z & s \end{bmatrix} \tag{63}$$

The maximization of $g(\bar{q})$ leads to the familiar eigenvalue problem

$$K\bar{q}^* = \lambda_{\max}\bar{q}^* \tag{64}$$

but in Flatland $\lambda_{\max}$ can be calculated in closed-form as

$$\lambda_{\max} = \sqrt{s^2 + z^2} \tag{65}$$

and

$$\bar{q}^* = \frac{1}{\sqrt{z^2 + (\lambda_{\max} + s)^2}} \begin{bmatrix} z \\ \lambda_{\max} + s \end{bmatrix} \tag{66}$$

The attitude variance of the BEST algorithm is calculated most easily from the Fisher information matrix using the fact that the BEST algorithm is a maximum-likelihood estimator [12]. Assuming the errors to have a Gaussian distribution, the calculation is straightforward and leads to

$$P^{\mathrm{BEST}} = \left[\sum_{k=1}^{n} \frac{1}{\sigma_{W_k}^2} \right]^{-1} \equiv \sigma_{\mathrm{tot}}^2 \tag{67}$$

The optimal angle of rotation can also be computed directly by noting that the gain function can be written in the form

$$g(\theta) = s \cos\theta + z \sin\theta \tag{68}$$

which is obviously a maximum when $\theta = \theta^*$, with

$$\cos\theta^* = \frac{s}{\sqrt{s^2 + z^2}} \quad \text{and} \quad \sin\theta^* = \frac{z}{\sqrt{s^2 + z^2}} \tag{69}$$

We write the solution of equation (69) more conveniently as

$$\theta^* = \arctan_2(z, s) \tag{70}$$

where $\arctan_2$ is the same function as ATAN2 in FORTRAN. Equation (69) leads directly to a solution for the optimal attitude matrix, namely

$$A^* = \frac{1}{\sqrt{s^2 + z^2}} \begin{bmatrix} s & z \\ -z & s \end{bmatrix} \tag{71}$$

Substitution of equation (66) into equation (17) leads somewhat less directly to the same result, which should be compared with the construction of the optimal attitude matrix in Space developed by Markley [13]. Markley's FOAM algorithm [14] carries over with little change into Flatland and yields necessarily the same result as equation (71). BEST (created in 1992) actually presages the algorithms of Mortari [15, 16], in that it computes the overlap eigenvalue $\lambda_{\max}$ from an analytical expression.[6]

The OIVAE Algorithm

> *It is high time that I should pass from these brief and discursive notes about things in Flatland to the central event …*

In 1992, when the preceding text was first presented [6], the specific large application which followed was the study of the "additive" and "multiplicative" updates in the Kalman filter, a subject of some controversy at the time. It would no

[6]For a masterly survey of solutions to the Wahba problem in Space the reader is directed to the paper of Markley and Mortari [17].

longer be appropriate to present that example in an archival journal, since a fuller account of the study in “Space” has already appeared [18–20]. Therefore, in order to illustrate the power of working in Flatland and the insights to be gained therefrom we examine instead a more recent application, the Optimal Linear Attitude Estimator (OLAE[7]) of Mortari, Markley, and Junkins [21]. That work contained some minor defects, chiefly in testing the algorithms. Firstly, the modeling of a uniformly random attitude followed what may be termed the “common wisdom” rather than sampling from a truly uniformly random attitude distribution [22–24]. Secondly, the figure of merit chosen to judge the efficacy of the algorithm was based on the farthest outlier and therefore diverges as the number of simulated tests become infinite. Both of these defects are inconsequential for the OLAE algorithm and easily repaired. On the other hand, the attitude estimator proposed was remarkably simple, original, and inventive. Missing from the presentation of that algorithm, however, was an analytical study of its covariance. The present work seeks now to remove that lacuna, at least in Flatland.

Let us now examine the Optimal Ingenious Vainglorious Attitude Estimator (OIVAE[8]), the Flatland counterpart of OLAE. In Space and in Flatland we may write

$$\begin{aligned}\hat{\mathbf{W}}_k &= A\hat{\mathbf{V}}_k + \Delta\hat{\mathbf{W}}_k \\ &= \frac{I + [\![\mathbf{g}]\!]}{I - [\![\mathbf{g}]\!]}\hat{\mathbf{V}}_k + \Delta\hat{\mathbf{W}}_k \end{aligned} \tag{72}$$

where $\mathbf{g}$ is the Gibbs vector in Space and the Gibbs scalar in Flatland (where, to be rigorous, it should not be written in boldface). We can write this equivalently as

$$(I - [\![\mathbf{g}]\!])\hat{\mathbf{W}}_k = (I + [\![\mathbf{g}]\!])\hat{\mathbf{V}}_k + (I - [\![\mathbf{g}]\!])\Delta\hat{\mathbf{W}}_k \tag{73}$$

which can be rearranged as

$$\hat{\mathbf{W}}_k - \hat{\mathbf{V}}_k = [\![\mathbf{g}]\!](\hat{\mathbf{W}}_k + \hat{\mathbf{V}}_k) + (I - [\![\mathbf{g}]\!])\Delta\hat{\mathbf{W}}_k \tag{74}$$

This suggests that we obtain the optimal attitude from the minimization of

$$J(\mathbf{g}) = \frac{1}{2}\sum_{k=1}^{n} a_k|\hat{\mathbf{W}}_k - \hat{\mathbf{V}}_k - [\![\mathbf{g}]\!](\hat{\mathbf{W}}_k + \hat{\mathbf{V}}_k)|^2 \tag{75}$$

with the optimal choice of the a_k, $k = 1, \ldots, n$, still to be determined. The minimization of the cost function of equation (75) is trivial both in Space and in Flatland. In Space this minimization yields the OLAE algorithm; in Flatland it yields OIVAE. Regrettably, the authors of reference [18] did not provide a model covariance matrix for OLAE. We now develop the covariance matrix (really variance) for OIVAE.

Equation (75) is remarkable in that without any geometric approximation, we have achieved a cost function which is linear in an unconstrained representation of the attitude. What is also special about OLAE is that the measurement noise also depends on the attitude. In fact, for angles of rotation close to π the measurement covariance is infinite. Thus, we might expect that OLAE may exhibit unpleasant behavior for very large angles.

[7]Pronounced “olay” as in Spanish: *¡olé!*.
[8]Pronounced “oy vay” as in Yiddish: “אוי וויי איז מיר!”

For OIVAE equation (75) is equivalently in Flatland

$$J(g) = \sum_{k=1}^{n} a_k |\hat{\mathbf{W}}_k - \hat{\mathbf{V}}_k - g\mathbf{J}(\hat{\mathbf{W}}_k + \hat{\mathbf{V}}_k)|^2 \tag{75$'$}$$

with the 2×2 matrix $\mathbf{J}$ defined in equation (9). Minimization of equation (75′) leads straightforwardly to

$$g^* = \frac{\langle \hat{\mathbf{W}}_k \times \hat{\mathbf{V}}_k \rangle}{1 + \langle \hat{\mathbf{W}}_k \cdot \hat{\mathbf{V}}_k \rangle} \tag{76}$$

where g^* is the value of g which minimizes $J(g)$. Here we have defined the notation

$$\langle F_k \rangle \equiv \sum_{k=1}^{n} a_k F_k \tag{77}$$

assuming, as usual, that the a_k, $k = 1, \ldots, n$, have unit sum.

To begin the construction of a covariance matrix for OIVAE we write

$$g^* = \delta g \circ g^{\text{true}} \tag{78}$$

where δg is the error Gibbs scalar and g^{true} is the true value of g. Then, recalling the composition rule for the Gibbs scalar, we have

$$g^* = \frac{g^{\text{true}} + \delta g}{1 - g^{\text{true}}\delta g} \tag{79}$$

and since δg is assumed to be much smaller than g^{true} we may expand equation (79) to first order in δg as

$$g^* = g^{\text{true}} + (1 + |g^{\text{true}}|^2)\,\delta g \tag{80}$$

expanding the right member of equation (76) to linear order in $\Delta\hat{\mathbf{W}}_k$ then leads to

$$(1 + |g^{\text{true}}|^2)\,\delta g = g^{\text{true}} \left[\frac{\langle \Delta\hat{\mathbf{W}}_k \times \hat{\mathbf{V}}_k \rangle}{\langle \hat{\mathbf{W}}_k^{\text{true}} \times \hat{\mathbf{V}}_k \rangle} - \frac{\langle \Delta\hat{\mathbf{W}}_k \cdot \hat{\mathbf{V}}_k \rangle}{1 + \langle \hat{\mathbf{W}}_k^{\text{true}} \cdot \hat{\mathbf{V}}_k \rangle} \right] \tag{81}$$

As usual, we assume for the sake of argument that the error in $\hat{\mathbf{V}}_k$ may be neglected in comparison with that in $\hat{\mathbf{W}}_k$, although this need not be the case.

Let θ be the (true) angle of rotation. Then in Flatland (but not in Space) we must have

$$\langle \hat{\mathbf{W}}_k^{\text{true}} \cdot \hat{\mathbf{V}}_k \rangle = \cos\theta \quad \text{and} \quad \langle \hat{\mathbf{W}}_k^{\text{true}} \times \hat{\mathbf{V}}_k \rangle = \sin\theta \tag{82ab}$$

and

$$\frac{g^{\text{true}}}{1 + |g^{\text{true}}|^2} = \frac{1}{2}\sin\theta \tag{83}$$

Note that δg is the Gibbs scalar of an infinitesimal rotation. Hence, it is related to the attitude error angle $\Delta\xi$ of equation (40) by

$$\delta g = \Delta\xi/2 \tag{84}$$

Combining equations (81) through (84) now leads to

$$\Delta\xi = \sin\theta\left[\frac{\langle \hat{\mathbf{W}}_k \times \Delta\hat{\mathbf{V}}_k\rangle}{\sin\theta} - \frac{\langle \hat{\mathbf{W}}_k \cdot \Delta\hat{\mathbf{V}}_k\rangle}{1+\cos\theta}\right] \tag{85}$$

It follows now that the variance of $\Delta\xi$ is given by

$$E\{|\Delta\xi|^2\} = \sin^2\theta\left[\frac{E\{\langle \hat{\mathbf{W}}_k \times \Delta\hat{\mathbf{V}}_k\rangle^2\}}{\sin^2\theta} - 2\,\frac{E\{\langle \hat{\mathbf{W}}_k \times \Delta\hat{\mathbf{V}}_k\rangle\langle \hat{\mathbf{W}}_k \cdot \Delta\hat{\mathbf{V}}_k\rangle\}}{\sin\theta(1+\cos\theta)} + \frac{E\{\langle \hat{\mathbf{W}}_k \cdot \Delta\hat{\mathbf{V}}_k\rangle^2\}}{(1+\cos\theta)^2}\right] \tag{86}$$

Assuming the BEST measurement model, which was given by equations (42) through (44) and especially by equation (47), we find straightforwardly

$$E\{\langle \hat{\mathbf{W}}_k \times \Delta\hat{\mathbf{V}}_k\rangle^2\} = \cos^2\theta \sum_{k=1}^{n} a_k^2\sigma_k^2 \tag{87a}$$

$$E\{\langle \hat{\mathbf{W}}_k \times \Delta\hat{\mathbf{V}}_k\rangle\langle \hat{\mathbf{W}}_k \cdot \Delta\hat{\mathbf{V}}_k\rangle\} = -\sin\theta\cos\theta \sum_{k=1}^{n} a_k^2\sigma_k^2 \tag{87b}$$

$$E\{\langle \hat{\mathbf{W}}_k \cdot \Delta\hat{\mathbf{V}}_k\rangle^2\} = \sin^2\theta \sum_{k=1}^{n} a_k^2\sigma_k^2 \tag{87c}$$

We have assumed in reaching equations (87) that the $\Delta\hat{\mathbf{W}}_k$ are a white sequence. Thus

$$E\{|\Delta\xi|^2\} = \sin^2\theta\left[\frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{1+\cos\theta}\right]^2 \sum_{k=1}^{n} a_k^2\sigma_k^2 \tag{88a}$$

$$= \sum_{k=1}^{n} a_k^2\sigma_k^2 \tag{88b}$$

With the a_k constrained to have unit sum, the minimum value for the summation in equation (88) (minimized over the a_k) is

$$\sum_{k=1}^{n} a_k^2\sigma_k^2 = \sigma_{\text{tot}}^2 \tag{89}$$

with $a_k = 1/\sigma_k^2$ and with σ_{tot}^2 defined in equation (67). Thus, finally

$$P^{\text{OIVAE}} = E\{|\Delta\xi|^2\} = \sigma_{\text{tot}}^2 = P^{\text{BEST}} \tag{90}$$

Discussion

… my Lord has shewn me the intestines of all my countrymen in the Land of Two Dimensions …

The representation of attitude in two dimensions has been described in detail. Two-dimensional analogues have been presented for the well known TRIAD and QUEST algorithms. The writer hopes that he has demonstrated the efficacy of studying attitude algorithms in only two dimensions, where closed-form expressions are usually available for the results, rather than insisting always on working in three dimensions, where a closed-form solution is usually horribly difficult or impossible to obtain.

The unexpected result of this work is that OIVAE performs as well as BEST, from which it might be inferred that OLAE performs as well as QUEST. This has not been demonstrated, however, and a full three-dimensional study is now called for. Had OIVAE shown a larger estimate error variance for large angles, or even diverged as the angle of rotation approached π, this writer would have been better satisfied. Not only would the value of Flatland studies have been bolstered, but one would then have abandoned OLAE, whereupon the writer, with unbearable smugness, would have boasted to OLAE's supposedly crestfallen creators of the clear superiority of QUEST. Instead, he finds that OLAE merits further attention.

A serious deficiency of the OLAE algorithm is that there is not as yet an obvious figure of merit, like TASTE in QUEST, for quickly judging the goodness of fit and setting the alarm for outliers in the data. (Ahah! QUEST and the other algorithms based on Davenport's q-method [8, 11], some even by OLAE's creators [13–16], *are* superior!) For actual mission operations, such a figure of merit is a more important attribute of an attitude determination algorithm than the speed of attitude computation [25]. As it stands, the only figure of merit suggested by the problem is $J(\mathbf{g}^*)$, the value of the cost function for the optimal Gibbs vector, whose interpretation is not obvious, owing to the presence of the factors $(I - [\![\mathbf{g}]\!])$ in the noise term. One could, of course, just as easily calculate the value of the Wahba cost function, but that still imposes a large computational burden. Of equal concern is the apparent absence of an easily calculable indicator for OLAE in order to know when one is near rotations of π, so that the method of sequential rotations [26] can be invoked if needed. Despite possible drawbacks of the algorithm, OLAE certainly merits further study.

Attitude studies in Flatland are by no means the innovation of the present writer, although he is almost certainly the first person to call it thus in print. In Spacecraft Attitude Control, for example, single-axis attitude control laws can be found even in undergraduate textbooks [27]. In Spacecraft Attitude Estimation Farrenkopf's celebrated single-axis study of steady-state Kalman filter performance with gyros [28] (updated recently by Markley and Reynolds [29]) is now a quarter century old. Still, the study of attitude estimation algorithms via two-dimensional analogies does not seem to have received the attention it deserves. Perhaps, the time has come for a blossoming of studies of Planecraft Attitude Estimation.

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"*You see … how little your words have done*."

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Appendix A: Properties of the Binion Propagation Matrix

Let **a** and **b** be the representations of any two 2×1 arrays and $\bar{a}$ and $\bar{b}$ the respective binions.

$$\{\bar{a}\}\{\bar{b}\} = \{\bar{b}\}\{\bar{a}\} = \{\bar{a} \otimes \bar{b}\} \tag{A1a}$$

$$\{\bar{a}\}\mathbf{b} = \{\bar{b}\}\mathbf{a} \tag{A1b}$$

$$\{\bar{a}\}^{\mathrm{T}}\{\bar{a}\} = |\mathbf{a}|^2\mathbf{I} \tag{A1c}$$

$$\{\bar{a}\}\{\bar{b}\}^{\mathrm{T}} = \mathbf{a} \cdot \mathbf{b}\,\mathbf{I} + \mathbf{a} \times \mathbf{b}\,\mathbf{J} \tag{A1d}$$

$$\{\bar{a}\}^{\mathrm{T}} = \{\tilde{\bar{a}}\} \quad \text{with} \quad \tilde{\bar{a}} = [-a_1, a_2]^{\mathrm{T}} \tag{A1e}$$

$$\{\bar{a}\}^{\mathrm{T}}\mathbf{b} = [\mathbf{b} \times \mathbf{a}, \mathbf{b} \cdot \mathbf{a}]^{\mathrm{T}} \tag{A1f}$$

For a binion $\bar{a}$

$$\bar{a}^{-1} = |\bar{a}|^{-2}\tilde{\bar{a}} \tag{A2}$$

It is useful also to define

$$\bar{a}_{\perp} \equiv \mathbf{J}\bar{a} = \begin{bmatrix} a_2 \\ -a_1 \end{bmatrix} \tag{A3}$$

Hence

$$\bar{a}_{\perp} \cdot \bar{a} = 0 \quad \text{and} \quad \bar{a}_{\perp} \times \bar{a} = -|\mathbf{a}|^2 \tag{A4ab}$$