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TWOSTEP: A Fast Robust Algorithm for Attitude-Independent Magnetometer-Bias Determination

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Abstract

A fast robust algorithm is developed for the inflight estimation of magnetometer biases when the attitude is not known. This algorithm combines the convergence in a single step of an heuristic algorithm currently in use with the correct treatment of the statistics of the measurement, and does this without discarding data. The new algorithm works well even when the magnetometer bias is comparable in magnitude to the ambient magnetic field. The algorithm performance is examined using simulated data for both spinning and inertially stabilized spacecraft.

Introduction

At orbit injection, often the only attitude sensor operating is the vector magnetometer. Frequently, the spacecraft is spinning rapidly, and, if the spacecraft is not in an equatorial orbit and not at too high an altitude, it is possible on the basis of this sensor alone (and, of course, a knowledge of the spacecraft position) to determine the spin rate and the spin-axis attitude of the spacecraft. At the same time, the accuracy of the magnetometer data may be compromised by large systematic magnetic disturbances on the spacecraft, often the result of space charging during launch or from electrical currents within the spacecraft. Thus, some means is usually needed to determine this bias quickly. Since the three-axis attitude of the spacecraft usually cannot be determined at this stage, the desired algorithm must not require a knowledge of the attitude as input.

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The above situation occurs for nearly every spacecraft. For spacecraft equipped with only a vector magnetometer and a Sun sensor, three-axis attitude will rely on the magnetometer data. In this case, the spacecraft attitude cannot be used directly to determine the magnetometer bias vector by transforming the reference magnetic field to magnetometer coordinates using the computed attitude and then comparing this transformed reference field with the magnetometer measurement. For such a mission, which occurs quite often, algorithms of the type discussed in this paper are required.[3]

Our study focuses on near-Earth spacecraft, for which a reasonably accurate magnetic field model exists (certainly for magnetic latitudes of less than 70 deg). Obviously, for scientific studies in which one wished to refine the geomagnetic field model, one would require complete attitude knowledge, at least as a practical matter. The algorithm studied in this work is adequate for calibrating parts of the attitude control system, such as for determining the ambient magnetic field for momentum dumping. For a spacecraft with moderate attitude accuracy requirements (say, approximately 1 deg/axis), which is almost always the case for spacecraft at orbit injection, it is adequate for calibrating the magnetometer for the attitude determination system as well.

A number of algorithms have been proposed for estimating the magnetometer bias. The simplest is to solve for the bias vector by minimizing the weighted sum of the squares of residuals which are the differences in the squares of the magnitudes of the measured and modeled magnetic fields [1]. Unfortunately, this leads to a cost function which is quartic in the magnetometer bias vector. To avoid the naive minimization of a quartic function of the magnetometer bias, a number of alternative methods have been proposed. These comprise the centered algorithm of Gambhir [1, 2], Davenport's quadratic approximation [3], and Acuña's model-independent method [4]. The new method, which we call TWOSTEP, is an improvement and considerable extension of Gambhir's algorithm. Gambhir's algorithm did not treat properly the correlations introduced by the centering process, nor did it attempt to correct for the possibly significant amount of data which the centering process discards. The new algorithm suffers from neither of these drawbacks and is very robust and efficient as well. The present paper presents the development of this new algorithm. The comparison with other methods will be carried out in a succeeding work [5].

The Measurement Model

We begin with the model

$$\mathbf{B}_k = A_k\mathbf{H}_k + \mathbf{b} + \boldsymbol{\varepsilon}_k, \qquad k = 1, \ldots, N \tag{1}$$

where $\mathbf{B}_k$ is the measurement of the magnetic field (more exactly, magnetic induction) by the magnetometer at time t_k; $\mathbf{H}_k$ is the corresponding value of the geomagnetic field with respect to an Earth-fixed coordinate system; A_k is the attitude of the magnetometer with respect to the Earth-fixed coordinates; $\mathbf{b}$ is the magnetometer bias; and $\boldsymbol{\varepsilon}_k$ is the measurement noise. The measurement noise, which includes both sensor errors and geomagnetic field model uncertainties, is generally assumed to be white and Gaussian. This is probably a poor approximation, since the errors in the

[3]One could also treat both the attitude and the magnetometer-bias vector in a Kalman filter. That approach adds significant complexity to the computations and may not yield a better result than the method developed here.

geomagnetic field model are certainly correlated, and, in fact, generally dominate the instrument errors. However, for the sake of argument we shall assume here that the errors are white and Gaussian.

To eliminate the dependence on the attitude, we transpose terms in equation (1) and compute the square, so that at each time

$$|\mathbf{H}_k|^2 = |A_k\mathbf{H}_k|^2 = |\mathbf{B}_k - \mathbf{b} - \boldsymbol{\varepsilon}_k|^2 \tag{2a}$$

$$= |\mathbf{B}_k|^2 - 2\mathbf{B}_k \cdot \mathbf{b} + |\mathbf{b}|^2 - 2(\mathbf{B}_k - \mathbf{b}) \cdot \boldsymbol{\varepsilon}_k + |\boldsymbol{\varepsilon}_k|^2 \tag{2b}$$

If we now define effective measurements and measurement noise according to

$$z_k \equiv |\mathbf{B}_k|^2 - |\mathbf{H}_k|^2 \tag{3a}$$

$$\nu_k \equiv 2(\mathbf{B}_k - \mathbf{b}) \cdot \boldsymbol{\varepsilon}_k - |\boldsymbol{\varepsilon}_k|^2 \tag{3b}$$

then we can write

$$z_k = 2\mathbf{B}_k \cdot \mathbf{b} - |\mathbf{b}|^2 + \nu_k, \qquad k = 1, \ldots, N \tag{4}$$

Thus, even with the assumption that the original magnetometer measurement noise is white and Gaussian with covariance matrix Σ_k, the effective measurement noise is not exactly white or Gaussian and will contain both Gaussian and χ^2 components. Thus, if

$$\boldsymbol{\varepsilon}_k \sim \mathcal{N}(\mathbf{0}, \Sigma_k) \tag{5}$$

and

$$E\{\boldsymbol{\varepsilon}_k \boldsymbol{\varepsilon}_\ell^{\mathrm{T}}\} = 0 \quad \text{for} \quad k \neq \ell \tag{6}$$

where $E\{\cdot\}$ denotes the expectation, it follows that

$$\mu_k \equiv E\{\nu_k\} = -\mathrm{tr}(\Sigma_k) \tag{7a}$$

$$\sigma_k^2 \equiv E\{\nu_k^2\} - \mu_k^2 = 4(\mathbf{B}_k - \mathbf{b})^{\mathrm{T}}\Sigma_k(\mathbf{B}_k - \mathbf{b}) + 2(\mathrm{tr}\,\Sigma_k^2) \tag{7b}$$

Here $\mathrm{tr}(\cdot)$ denotes the trace operation. In addition,

$$E\{\nu_k \nu_\ell) = \mu_k \mu_\ell \quad \text{for} \quad k \neq \ell \tag{8}$$

so that the $\nu_k - \mu_k$ are white. Generally, we assume that the noise $\boldsymbol{\varepsilon}_k$ is small compared to the geomagnetic field, which is certainly true in low-Earth orbit. Then to a large degree ν_k is Gaussian and we can write approximately

$$\nu_k \sim \mathcal{N}(\mu_k, \sigma_k^2) \tag{9}$$

Note that $\mathbf{B}_k$ in equation (3b) is a random variable, being equal to $\mathbf{B}_k^{\mathrm{true}} + \boldsymbol{\varepsilon}_k$. Thus, we could rewrite equation (4) more correctly as

$$z_k = 2\mathbf{B}_k^{\mathrm{true}} \cdot \mathbf{b} - |\mathbf{b}|^2 + u_k, \qquad k = 1, \ldots, N \tag{4$'$}$$

with

$$u_k \equiv 2(\mathbf{B}_k^{\mathrm{true}} - \mathbf{b}) \cdot \boldsymbol{\varepsilon}_k + |\boldsymbol{\varepsilon}_k|^2 \tag{3b$'$}$$

However, we cannot know $\mathbf{B}_k^{\mathrm{true}}$ without calculating A_k, which we wish specifically to avoid. Hence, we must fall back on equations (3b) and (7) as the governing equations of our effective measurement model. In general, the error introduced in this way will be very small, because $\mu_k^2 \ll \sigma_k^2$, by nearly four orders of magnitude, in fact, if we do not exceed 70 deg in magnetic latitude.

Scoring

Given the statistical model above, the negative-log-likelihood function [6] for the magnetometer bias is given by

$$J(\mathbf{b}) = \frac{1}{2}\sum_{k=1}^{N}\left[\frac{1}{\sigma_k^2}(z'_k - 2\mathbf{B}_k \cdot \mathbf{b} + |\mathbf{b}|^2 - \mu_k)^2 + \log \sigma_k^2 + \log 2\pi\right] \quad (10)$$

which is quartic in $\mathbf{b}$. Here z'_k denotes the realization of z_k, i.e. the actual value obtained in the data, to distinguish it from the random variable z_k. A prime will be used to designate the realization of a random variable throughout this paper. This rule will not apply to the $\mathbf{B}_k$, because these always denote the value in the data.

The maximum-likelihood estimate maximizes the likelihood of the estimate of the bias, which is the probability density of the measurements (evaluated at their sampled values) given as a function of the magnetometer bias. Hence, it minimizes the negative logarithm of the likelihood (equation (10)), which thus provides a cost function.

Since the domain of J has no boundaries, the maximum-likelihood estimate for $\mathbf{b}$, which we denote by $\mathbf{b}^*$, must satisfy

$$\left.\frac{\partial J}{\partial \mathbf{b}}\right|_{\mathbf{b}^*} = \mathbf{0} \quad (11)$$

Note that only the first of the three terms under the summation depends on the magnetometer bias. Unless one wishes to estimate parameters of the measurement noise, there is no reason to retain the remaining two terms.[4] This quartic dependence can be avoided if complete three-axis attitude information is available, since the bias term then enters linearly into the measurement model (q.v. equation (1)) as in the work of Lerner and Shuster [7].

The most direct solution is obtained by scoring, which in this case is the Newton–Raphson approximation. Since an *a priori* estimate of the magnetometer bias is generally not available, we consider the sequence[5]

$$\mathbf{b}_0^{NR} = \mathbf{0} \quad (12a)$$

$$\mathbf{b}_{i+1}^{NR} = \mathbf{b}_i^{NR} = \left[\frac{\partial^2 J}{\partial \mathbf{b} \partial \mathbf{b}^T}(\mathbf{b}_i^{NR})\right]^{-1} \frac{\partial J}{\partial \mathbf{b}}(\mathbf{b}_i^{NR}) \qquad i = 0, 1, \ldots. \quad (12b)$$

This sequence is obtained by expanding $J(\mathbf{b})$ to quadratic order in $(\mathbf{b} - \mathbf{b}_i^{NR})$, setting the gradient of the truncated series to zero, and solving for $\mathbf{b}_{i+1}$. If for some value of i we are sufficiently close to the maximum-likelihood estimate, then as i tends to infinity, $\mathbf{b}_i^{NR}$ will tend toward a minimum or maximum of $J(\mathbf{b})$. Unfortunately, the quartic nature of $J(\mathbf{b})$ leads to multiple minima and maxima so that the convergence to the desired global minimum is by no means guaranteed.

A modification of equations (12) in frequent use is to replace the Hessian matrix (the matrix of second partial derivatives) of $J(\mathbf{b})$, by its expectation value, the Fisher information matrix F_{bb}. Under not very restrictive conditions, as the amount of data becomes infinite (or for even small samples for Gaussian measurement noise, as as-

[4]In fact, the standard deviations do depend on the bias vector as shown by equation (7b). However, we take the point of view that the standard deviations are functions of the true value of the bias vector. The dependence of the estimate of the bias vector on the weights is not very strong in any event.

[5]Throughout this work we shall use k as the time index and i as the iteration index.

sumed here), the estimate error covariance matrix P_{bb} becomes the inverse of the Fisher information matrix. The procedure of replacing the Hessian matrix by the Fisher information matrix, called the Gauss-Newton method, usually results in some simplification through the discarding of complicated terms with vanishing mean, but does not address the problem of multiple critical values.

The Centered Estimate

In order to avoid the minimization of a quartic cost function, let us define in a manner similar to Gambhir [1, 2] the following weighted averages

$$\bar{z} \equiv \bar{\sigma}^2 \sum_{k=1}^{N} \frac{1}{\sigma_k^2} z_k, \qquad \bar{\mathbf{B}} \equiv \bar{\sigma}^2 \sum_{k=1}^{N} \frac{1}{\sigma_k^2} \mathbf{B}_k \tag{13ab}$$

$$\bar{\nu} \equiv \bar{\sigma}^2 \sum_{k=1}^{N} \frac{1}{\sigma_k^2} \nu_k, \qquad \bar{\mu} \equiv \bar{\sigma}^2 \sum_{k=1}^{N} \frac{1}{\sigma_k^2} \mu_k \tag{13cd}$$

where

$$\frac{1}{\bar{\sigma}^2} \equiv \sum_{k=1}^{N} \frac{1}{\sigma_k^2} \tag{14}$$

Then it follows that

$$\bar{z} = 2\bar{\mathbf{B}} \cdot \mathbf{b} - |\mathbf{b}|^2 + \bar{\nu} \tag{15}$$

If we define now

$$\tilde{z}_k \equiv z_k - \bar{z}, \qquad \tilde{\mathbf{B}}_k \equiv \mathbf{B}_k - \bar{\mathbf{B}} \tag{16ab}$$

$$\tilde{\nu}_k \equiv \nu_k - \bar{\nu}, \qquad \tilde{\mu}_k \equiv \mu_k - \bar{\mu} \tag{16cd}$$

then subtracting equation (15) from equation (4) leads to

$$\tilde{z}_k = 2\tilde{\mathbf{B}}_k \cdot \mathbf{b} + \tilde{\nu}_k, \qquad k = 1, \ldots, N \tag{17}$$

This operation is called *centering*.

The centered measurements, equation (17), are no longer quadratic in the magnetometer bias vector, so that using the centered measurements alone we can solve for $\mathbf{b}^*$ in a single iteration of the Newton–Raphson or Gauss-Newton method. However, the centered measurement noise is no longer uncorrelated. Thus, one can no longer write the negative-log-likelihood function in the form of equation (10), that is, as the sum of N squares. Nonetheless, in practice one has generally ignored this correlation and determined the bias from an approximate cost function of the form

$$J^{\text{approx}}(\mathbf{b}) = \frac{1}{2} \sum_{k=1}^{N-1} \frac{1}{\sigma_k^2} (\tilde{z}'_k - 2\tilde{\mathbf{B}}_k \cdot \mathbf{b} - \tilde{\mu}_k)^2 \tag{18}$$

and achieved reasonable results in spite of the lack of mathematical consistency and rigor, arguing that one was only discarding a single measurement out of many. In actual practice, these calculations have usually assumed a constant weighting and neglected the contribution of $\tilde{\mu}_k$. Gambhir's RESIDG algorithm [2], however, is presented with variable weights, although the correlations are not treated correctly, and it was assumed that $\tilde{\mu}_k = 0$. In addition, Gambhir included all N measurements without justification, since they are not independent.

We shall see below that one can discard much more than $1/N$ of the accuracy by this operation, but we shall see also that equation (18) is closer to being correct than one might have imagined. Note that the sum is from 1 to $N - 1$, since the centered measurements are not independent.

$$\sum_{k=1}^{N} \frac{1}{\sigma_k^2} \tilde{z}_k = 0 \tag{19}$$

Minimizing $J^{\text{approx}}(\mathbf{b})$ over $\mathbf{b}$ leads to

$$\mathbf{b}^{*\text{approx}} = P_{bb}^{\text{approx}} \sum_{k=1}^{N-1} \frac{1}{\sigma_k^2} (\tilde{z}'_k - \tilde{\mu}_k) 2\tilde{\mathbf{B}}_k \tag{20}$$

with the estimate error covariance matrix given approximately by

$$P_{bb}^{\text{approx}} \approx (F_{bb}^{\text{approx}})^{-1} = \left[\sum_{k=1}^{N-1} \frac{1}{\sigma_k^2} 4\tilde{\mathbf{B}}_k\tilde{\mathbf{B}}_k^{\mathrm{T}} \right]^{-1} \tag{21}$$

Note that $\tilde{\mu}_k$ will vanish if the original measurement noise $\boldsymbol{\varepsilon}_k$, $k = 1, \ldots, N$, is identically distributed. The approximate centered estimator converges in a single iteration because the cost function is exactly quadratic. However, equations (20) and (21) rest on incorrect statistical assumptions.

A Statistically Correct Centered Algorithm

The original measurements, z_k, $k = 1, \ldots, N$, may be replaced by the centered measurements, $\tilde{z}_k$, $k = 1, \ldots, N - 1$, and the center value $\bar{z}$, without loss of information. This follows from the fact that the centered measurements and center measurements are obtained from the original measurements by a nonsingular linear transformation. The measurement equations are given by equations (15) and (17). The centered data have the advantage of depending only linearly on the magnetometer bias. However, they have the disadvantage that the centered measurement noise is correlated. Therefore, the negative-log-likelihood function for the centered data alone cannot be written as the sum of $N - 1$ squares. To write a statistically correct cost function for the centered data (making the approximation that the measurement noise ν_k is Gaussian) we define

$$\tilde{\mathcal{Z}} \equiv [\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_{N-1}]^{\mathrm{T}}, \qquad \tilde{\mathcal{B}} \equiv [\tilde{\mathbf{B}}_1, \tilde{\mathbf{B}}_2, \ldots, \tilde{\mathbf{B}}_{N-1}]^{\mathrm{T}} \tag{22ab}$$

$$\tilde{\mathcal{M}} = [\tilde{\mu}_1, \tilde{\mu}_2, \ldots, \tilde{\mu}_{N-1}]^{\mathrm{T}}, \qquad \tilde{\mathcal{V}} \equiv [\tilde{\nu}_1, \tilde{\nu}_2, \ldots, \tilde{\nu}_{N-1}]^{\mathrm{T}} \tag{22cd}$$

and write

$$\tilde{\mathcal{Z}} = 2\tilde{\mathcal{B}}\mathbf{b} + \tilde{\mathcal{V}} \tag{23}$$

with

$$\tilde{\mathcal{V}} \sim \mathcal{N}(\tilde{\mathcal{M}}, \tilde{\mathcal{R}}) \tag{24}$$

Here $\tilde{\mathcal{R}}$ is the covariance matrix of $\tilde{\mathcal{V}}$, and $\tilde{\mathcal{M}}$ is the mean. The stacked measurement $\tilde{\mathcal{B}}$ is an $(N - 1) \times 3$ matrix, and $\tilde{\mathcal{R}}$ is an $(N - 1) \times (N - 1)$ positive-definite matrix whose elements are fully populated.

The negative-log-likelihood function for this stacked centered measurement is simply

$$\tilde{J}(\mathbf{b}) = \frac{1}{2}[(\tilde{\mathcal{Z}}' - 2\tilde{\mathcal{B}}\mathbf{b} - \tilde{\mathcal{M}})^{\mathrm{T}}\tilde{\mathcal{R}}^{-1}(\tilde{\mathcal{Z}}' - 2\tilde{\mathcal{B}}\mathbf{b} - \tilde{\mathcal{M}})$$
$$+ \log\det\tilde{\mathcal{R}} + (N-1)\log 2\pi] \tag{25}$$

Equation (18), because it neglects cross terms, expresses the incorrect assumption that $\tilde{\mathcal{R}}$ is diagonal. We do not make this approximation here. Minimizing the negative-log-likelihood function of equation (25) leads directly to the correctly centered estimate

$$\tilde{\mathbf{b}}^* = (4\tilde{\mathcal{B}}^{\mathrm{T}}\tilde{\mathcal{R}}^{-1}\tilde{\mathcal{B}})^{-1}2\tilde{\mathcal{B}}^{\mathrm{T}}\tilde{\mathcal{R}}^{-1}(\tilde{\mathcal{Z}} - \tilde{\mathcal{M}}) \tag{26}$$

with estimate error covariance matrix

$$\tilde{P}_{bb} = (4\tilde{\mathcal{B}}^{\mathrm{T}}\tilde{\mathcal{R}}^{-1}\tilde{\mathcal{B}})^{-1} \tag{27}$$

For large quantities of data, the naive evaluation of equations (26) and (27) can be a formidable task. Therefore, we seek the means of inverting the matrix in equation (25) explicitly. By direct substitution

$$\tilde{\mathcal{R}}_{k\ell} = E\{(\nu_k - \mu_k)(\nu_\ell - \mu_\ell) - (\nu_k - \mu_k)(\bar{\nu} - \bar{\mu})$$
$$- (\bar{\nu} - \bar{\mu})(\nu_\ell - \mu_\ell) + (\bar{\nu} - \bar{\mu})^2\} \tag{28}$$

It is a simple matter to show that

$$E\{(\nu_k - \mu_k)(\nu_\ell - \mu_\ell)\} = \sigma_k^2\,\delta_{k\ell} \tag{29a}$$

$$E\{(\nu_k - \mu_k)(\bar{\nu} - \bar{\mu})\} = E\{(\bar{\nu} - \bar{\mu})(\nu_\ell - \mu_\ell)\} = \bar{\sigma}^2 \tag{29b}$$

$$E\{(\bar{\nu} - \bar{\mu})^2\} = \bar{\sigma}^2 \tag{29c}$$

with $\delta_{k\ell}$ the usual Kronecker delta, which is equal to unity when the two indices are equal and zero otherwise.

Hence

$$\tilde{\mathcal{R}}_{k\ell} = \sigma_k^2\,\delta_{k\ell} - \bar{\sigma}^2 \tag{30}$$

which has the simple inverse

$$(\tilde{\mathcal{R}}^{-1})_{k\ell} = \frac{1}{\sigma_k^2}\delta_{k\ell} + \frac{\sigma_N^2}{\sigma_k^2\sigma_\ell^2} \tag{31}$$

where σ_N^2 is the variance of ν_N. Substituting this expression into equation (25) leads to

$$\tilde{J}(\mathbf{b}) = \frac{1}{2}\sum_{k=1}^{N}\frac{1}{\sigma_k^2}(\tilde{z}_k' - 2\tilde{\mathbf{B}}_k\cdot\mathbf{b} - \tilde{\mu}_k)^2 + \text{terms independent of } \mathbf{b} \tag{32}$$

The statistically correct cost function for the centered data looks exactly like the naive expression of equation (18) except that the summation is now from 1 to N, a truly remarkable result. The minimization is simple now and leads directly to

$$\tilde{\mathbf{b}}^* = \tilde{P}_{bb}\sum_{k=1}^{N}\frac{1}{\sigma_k^2}(\tilde{z}_k - \tilde{\mu}_k)2\tilde{\mathbf{B}}_k \tag{33}$$

and the estimate error covariance of the centered estimate is given by

$$\tilde{P}_{bb} = \tilde{F}_{bb}^{-1} = \left[\sum_{k=1}^{N} \frac{1}{\sigma_k^2} 4\tilde{\mathbf{B}}_k \tilde{\mathbf{B}}_k^{\mathrm{T}} \right]^{-1} \tag{34}$$

This correctly centered estimate is more attractive than the heuristic estimate of RESIDG. It is simple, and it treats the correlation of the centered measurement noise correctly. Although similar in form, it is very different in character from the centered estimate of Gambhir [1, 2]. The only drawback to the correctly centered algorithm lies in the exclusion of certain data, namely, the center term $\bar{z}$, the effect of which we investigate and eliminate in the next section.

We note in passing that the calculation of the remaining terms in equation (32) is not difficult. The result, which is developed in the appendix, is simply

$$\tilde{J}(\mathbf{b}) = \frac{1}{2} \left\{ \sum_{k=1}^{N} \left[\frac{1}{\sigma_k^2} (\tilde{z}_k' - 2\tilde{\mathbf{B}}_k \cdot \mathbf{b} - \tilde{\mu}_k)^2 + \log \sigma_k^2 + \log 2\pi \right] - [\log \bar{\sigma}^2 + \log 2\pi] \right\} \tag{35}$$

The Complete Solution with Correction for Centering

The rigorously centered algorithm derived above is no more complicated than the naive centered algorithm presented earlier. From the standpoint of computational burden, the more rigorous treatment of the statistics has merely added one more term (out of N) to the summation. However, equation (35), because it has been derived rigorously, affords us the possibility of computing the correction arising from the discarded center measurement $\bar{z}$. (Note the nomenclature: *center term* or *center measurement* for $\bar{z}$, *centered measurements* for the $\tilde{z}_k$, $k = 1, \ldots, N$.)

Instead of the measurement set $\{\tilde{z}_k, k = 1, \ldots, N-1; \bar{z}\}$, we may now consider the measurements to be effectively $\{\tilde{\mathbf{b}}^*, \bar{z}\}$, since for a linear Gaussian estimation problem, the maximum-likelihood estimate is a sufficient statistic [6], as we shall demonstrate explicitly below. Therefore, to determine the complete maximum likelihood estimate $\mathbf{b}^*$, we must develop the statistics of these two effective measurements more completely.

Note that up to now we have regarded $\mathbf{b}^*$ as the result of the estimation process depending only on the data. However, if we wish now to use the bias estimate as an effective measurement, then we must distinguish between the estimate $\mathbf{b}^{*\prime}$, i.e., the computed value of the magnetometer-bias vector, which depends only on the data $\{\mathbf{z}_k'; k = 1, \ldots, N\}$ and the estimator $\mathbf{b}^*$, (a random variable) which is the identical function of the random measurement variables $\{\mathbf{z}_k; k = 1, \ldots, N\}$. Henceforth, when a given relation is true *mutatis mutandis* for both the estimator and the estimate, and the content allows, the statement will be made for the estimator.

To see that $\tilde{\mathbf{b}}^*$ is a sufficient statistic for $\mathbf{b}$, substitute equation (17) into equation (33). This leads to

$$\tilde{\mathbf{b}}^* = \tilde{P}_{bb} \sum_{k=1}^{N} \frac{1}{\sigma_k^2} (2\tilde{\mathbf{B}}_k \cdot \mathbf{b} + \tilde{v}_k - \tilde{\mu}_k) 2\tilde{\mathbf{B}}_k \tag{36}$$

which we may rewrite as

$$\tilde{\mathbf{b}}^* = \mathbf{b} + \tilde{P}_{bb} \sum_{k=1}^{N} \frac{1}{\sigma_k^2} 2\tilde{\mathbf{B}}_k(\tilde{\nu}_k - \tilde{\mu}_k) \tag{37a}$$

$$\equiv \mathbf{b} + \tilde{\mathbf{v}}_b \tag{37b}$$

The last term is just the (zero-mean) estimate error. Obviously

$$\tilde{\mathbf{v}}_b \sim \mathcal{N}(\mathbf{0}, \tilde{P}_{bb}) \tag{38}$$

It follows that we can write

$$\tilde{J}(\mathbf{b}) = \frac{1}{2}(\mathbf{b} - \tilde{\mathbf{b}}^{*\prime})^T \tilde{P}_{bb}^{-1}(\mathbf{b} - \tilde{\mathbf{b}}^{*\prime}) + \text{terms independent of } \mathbf{b} \tag{39}$$

which can be verified by expanding equation (32) and completing the square in $\mathbf{b}$. But this is just the data-dependent term of the negative-log likelihood function of $\mathbf{b}$ given equations (37b) and (38). It is equation (39) which makes $\tilde{\mathbf{b}}^*$ a sufficient statistic for $\mathbf{b}$. Equation (39) is very useful, because it allows us to investigate the effect of corrections to the centered formula using only our knowledge of $\tilde{\mathbf{b}}^*$ and $\tilde{P}_{bb}$. We do not have to refer again to the N centered measurements $\tilde{z}_k$, $k = 1, \ldots, N$.

We must now combine $\tilde{\mathbf{b}}^*$ and $\bar{z}$ to obtain a complete representation of our data for the computation of $\mathbf{b}$. Recall equation (15)

$$\bar{z} = 2\overline{\mathbf{B}} \cdot \mathbf{b} - |\mathbf{b}|^2 + \bar{\nu} \tag{15}$$

with

$$\bar{\nu} \sim \mathcal{N}(\bar{\mu}, \bar{\sigma}^2) \tag{40}$$

Note that $\bar{z}$, which, unfortunately, is a nonlinear function of $\mathbf{b}$, is nonetheless an extremely accurate measurement, more accurate than the other measuresments by typically a factor of $1/\sqrt{N}$, because $\bar{\sigma}$ is smaller typically than the other variances by this factor. Thus, simply centering the data and discarding $\bar{z}$ can entail a significant loss of accuracy if the sensitivity of $\bar{z}$ to $\mathbf{b}$ is not small.

What is the correlation between $\tilde{\mathbf{v}}_b$ and $\bar{\nu}$? Calculating this explicitly, gives

$$E\{\tilde{\mathbf{v}}_b(\bar{\nu} - \bar{\mu})\} = 2\tilde{P}_{bb} \sum_{k=1}^{N} \frac{1}{\sigma_k^2} \tilde{\mathbf{B}}_k E\{(\tilde{\nu}_k - \tilde{\mu}_k)(\bar{\nu} - \bar{\mu})\} \tag{41a}$$

$$= 2\tilde{P}_{bb} \sum_{k=1}^{N} \frac{1}{\sigma_k^2} \tilde{\mathbf{B}}_k E\{(\nu_k - \mu_k)(\bar{\nu} - \bar{\mu}) - (\bar{\nu} - \bar{\mu})(\bar{\nu} - \bar{\mu})\} \tag{41b}$$

$$= 2\tilde{P}_{bb} \sum_{k=1}^{N} \frac{1}{\sigma_k^2} \tilde{\mathbf{B}}_k(\bar{\sigma}^2 - \bar{\sigma}^2) \tag{41c}$$

$$= \mathbf{0} \tag{41d}$$

Thus, $\tilde{\mathbf{v}}_b$ and $\bar{\nu}$ are uncorrelated. Since the measurement errors were assumed to be Gaussian, it follows that $\tilde{\mathbf{v}}_b$ and $\bar{\nu}$ are independent. The joint probability density function of $\tilde{\mathbf{v}}_b$ and $\bar{\nu}$ is therefore the product of the two individual probability density functions. Thus, the two corresponding negative-log-likelihood functions add

$$J(\mathbf{b}) = \tilde{J}(\mathbf{b}) + \bar{J}(\mathbf{b}) \tag{42}$$

with $\tilde{J}(\mathbf{b})$ given by equation (39) and

$$\bar{J}(\mathbf{b}) = \frac{1}{2}\left[\frac{1}{\bar{\sigma}^2}(\bar{z}' - 2\overline{\mathbf{B}}\cdot\mathbf{b} + |\mathbf{b}|^2 - \bar{\mu})^2 + \log\bar{\sigma}^2 + \log 2\pi\right] \tag{43}$$

The weight associated with the center term $\bar{J}(\mathbf{b})$ is equal to the sum of all the weights of $\tilde{J}(\mathbf{b})$. Thus, when $\overline{\mathbf{B}} - \mathbf{b}^{\text{true}}$ is not small, the loss of accuracy from discarding the center term can be substantial, as we shall see explicitly in some of the numerical examples. We can determine the relative importance of these terms to the estimate accuracy by computing the Fisher information matrix F_{bb} to obtain

$$F_{bb} = E\left\{\frac{\partial^2 J}{\partial\mathbf{b}\,\partial\mathbf{b}^{\text{T}}}\right\} = E\left\{\frac{\partial^2 \tilde{J}}{\partial\mathbf{b}\,\partial\mathbf{b}^{\text{T}}}\right\} + E\left\{\frac{\partial^2 \bar{J}}{\partial\mathbf{b}\,\partial\mathbf{b}^{\text{T}}}\right\} = \tilde{F}_{bb} + \bar{F}_{bb} \tag{44a}$$

$$= \tilde{P}_{bb}^{-1} + \frac{4}{\bar{\sigma}^2}(\overline{\mathbf{B}} - \mathbf{b})(\overline{\mathbf{B}} - \mathbf{b})^{\text{T}} \tag{44b}$$

$$= P_{bb}^{-1} \tag{44c}$$

The estimate error covariance matrix will be the inverse of this quantity. If the distribution of the magnetometer measurements is "isotropic," that is, if $\overline{\mathbf{B}} - \mathbf{b}^{\text{true}}$ vanishes, then $\bar{J}(\mathbf{b})$ will be insensitive to $\mathbf{b}$. It is in this case that the centering approximation obviously leads to the best results. If, however, one attempts to determine the magnetometer bias from a short data span, say, from an inertially stabilized or Earth-pointing spacecraft, then $\overline{\mathbf{B}} - \mathbf{b}^{\text{true}}$ will be equal to the similar expression for a typical value of the magnetic field, and the formerly discarded center term which will provide half or more of the accuracy, especially for the component along $\overline{\mathbf{B}} - \mathbf{b}^{\text{true}}$.

Because $\tilde{\mathbf{b}}^*$ provides a consistent estimator of the magnetometer bias vector, its errors are characterized by the Fisher information matrix, which can then be used to assess the need to compute the correction due to the discarded center term. If a diagonal element of the Fisher information $\bar{F}_{bb}$ of the center term alone computed at $\tilde{\mathbf{b}}^*$ is large compared to the corresponding element of $\tilde{F}_{bb}$ then we must compute the center correction. If it is much smaller in all three cases, the center term may be discarded without worry. We are thus led to the following two-step algorithm, which we call TWOSTEP:

- We compute the centered estimate $\tilde{\mathbf{b}}^{*\prime}$ of the magnetometer bias and the covariance matrix $\tilde{P}_{bb}$ using the centered data and equations (33) and (34).
- At the centered estimate $\tilde{\mathbf{b}}^{*\prime}$ we compute $\tilde{F}_{bb}$ and $\bar{F}_{bb}$ from equations (34) and (44b). If the diagonal elements of $\bar{F}_{bb}$ are sufficiently small compared with the corresponding elements of $\tilde{F}_{bb}$

$$[\bar{F}_{bb}]_{mm} < c[\tilde{F}_{bb}]_{mm}, \qquad m = 1, 2, 3 \tag{45}$$

  then we will terminate the computation of the magnetometer bias at the computation of $\tilde{\mathbf{b}}^*$ and accept this value as the estimate with the estimate error covariance matrix given by the inverse of $\tilde{F}_{bb}$. Otherwise,
- Using the centered estimate $\tilde{\mathbf{b}}^{*\prime}$ as an initial estimate, the correction due to the center term is computed using the Gauss–Newton method

$$\mathbf{b}_{i+1} = \mathbf{b}_i - F_{bb}^{-1}(\mathbf{b}_i)\mathbf{g}(\mathbf{b}_i) \tag{46}$$

where the Fisher information matrix $F_{bb}(\mathbf{b})$ is given by equation (44), and the gradient vector is given by the sum of the gradients of equations (39) and (43)

$$\mathbf{g}(\mathbf{b}) = \tilde{\mathbf{g}}(\mathbf{b}) + \bar{\mathbf{g}}(\mathbf{b})$$

$$= \tilde{P}_{bb}^{-1}(\mathbf{b} - \tilde{\mathbf{b}}^*) - \frac{1}{\bar{\sigma}^2}(\bar{z}' - 2\bar{\mathbf{B}} \cdot \mathbf{b} + |\mathbf{b}|^2 - \bar{\mu})2(\bar{\mathbf{B}} - \mathbf{b}) \qquad (47)$$

- The last step is iterated until

$$\eta_i \equiv (\mathbf{b}_i - \mathbf{b}_{i-1})^T F_{bb}(\mathbf{b}_{i-1})(\mathbf{b}_i - \mathbf{b}_{i-1}) \qquad (48)$$

is less than some predetermined small quantity.

Since the centered estimate was consistent, we expect that

$$\delta \equiv (\mathbf{b}^{*\prime} - \tilde{\mathbf{b}}^{*\prime})^T \tilde{P}_{bb}^{-1}(\mathbf{b}^{*\prime} - \tilde{\mathbf{b}}^{*\prime}) \qquad (49)$$

will not be large. If $\mathbf{b}^{*\prime}$ were the exact value of $\mathbf{b}$, then we should expect that this quantity would be $\chi^2(3)$, which has mean 3 and variance 6. The mean and variance of δ should be typically smaller than this. A large value of δ might indicate convergence to a non-global minimum of $J(\mathbf{b})$.

How large should c be in the test for computing the center correction, equation (45)? If we choose c to be 0.5, then the center correction will be computed only if it improves the accuracy by at least 20 percent. If we choose c to be 0.1, then the center correction will be computed only if it improves the accuracy by at least 5 percent. A reasonable value for c seems to be somewhere between these two numbers, depending on the taste of the user.

Numerical Examples

The new algorithm developed in this work has been examined for two typical scenarios: a spacecraft spinning at 15 rpm and an inertially stabilized spacecraft. The spacecraft orbit has been chosen to be circular with an altitude of 560 km and an inclination of 38 deg. The geomagnetic field in our studies has been simulated using the International Geomagnetic Reference Field (IGRF (1985)) [8], which has been extrapolated to 1994. More recent field models are available, but IGRF (1985) is adequate for our simulation needs.

For purposes of simulation we have assumed an effective white Gaussian magnetometer measurement error with isotropic error distribution and a standard deviation per axis of 2.0 mG, corresponding to an angular error of approximately 0.5 deg at the equator. We have assumed also that the x-axis of the magnetometer is parallel to the spacecraft spin axis, which always points toward the Sun. The Sun direction makes an angle of approximately 40 degrees with the orbit plane. Thus, for a spinning spacecraft we expect the estimation errors for the magnetometer bias to be largest for the x-component. The magnetometer data were sampled every eight seconds. All entries in the tables for the estimated magnetometer bias and the associated standard deviations are in mG.

Table 1 displays the results for the case of a spinning spacecraft and Table 2 for an inertially stabilized spacecraft. We have generally displayed all iterations up to convergence to two decimal places. The results are seen to be quite good in all cases. In only a few cases (in Table 3 below) were more than one iteration of the

TABLE 1. Performance of TWOSTEP for a Spinning Spacecraft

Step	Bias Estimate (mG)
$\mathbf{b}^{true} = [10., 20., 30.]$ mG	
centering approximation	[9.99, 20.10, 29.97] ±[0.07, 0.06, 0.11]
with center correction	[9.99, 20.10, 29.97] ±[0.07, 0.06, 0.11]
$\mathbf{b}^{true} = [100., 200., 300.]$ mG	
centering approximation	[100.17, 199.88, 299.94] ±[0.09, 0.08, 0.14]
with center correction	[100.17, 199.88, 299.94] ±[0.09, 0.08, 0.14]

center correction required to this accuracy. In most cases, the centering approximation alone was sufficient to this level of accuracy. Nearly 200 different cases were simulated in testing the algorithm. The above cases were typical except that for Table 2, we have modified the field model slightly so that the third component of the bias would be less observable from the centered data alone. This was done to illustrate more acutely the possible importance of the center correction.

In Table 1, the confidence intervals for the centered estimate are indistinguishable from those of the final result incorporating the center correction, and the estimates themselves are identical to two decimal places. As mentioned earlier, we can avoid an unnecessary computation of the center correction by examining the Fisher information matrices $\tilde{F}_{bb}$ and $\bar{F}_{bb}$ immediately following the computation of the center correction. This is well illustrated by the present examples. Consider the example of the spinning spacecraft in Table 1 for the small bias value. Here the Fisher information for the centered estimate $\tilde{F}_{bb}$, computed at $\tilde{\mathbf{b}}$ has the value

$$\tilde{F}_{bb} = \begin{bmatrix} 192.82 & -0.87 & 0.77 \\ -0.87 & 258.76 & 7.98 \\ 0.77 & 7.09 & 80.61 \end{bmatrix} (\text{mG})^{-2} \tag{50a}$$

TABLE 2. Performance of TWOSTEP for an Inertially Stabilized Spacecraft

Step	Bias Estimate (mG)
$\mathbf{b}^{true} = [10., 20., 30.]$ mG	
centering approximation	[9.92, 20.00, 29.68] ±[0.14, 0.33, 0.98]
with center correction	[9.94, 19.94, 29.92] ±[0.11, 0.17, 0.11]
$\mathbf{b}^{true} = [100., 200., 300.]$ mG	
centering approximation	[99.92, 200.01, 299.68] ±[0.14, 0.33, 0.98]
with center correction	[99.94, 199.94, 299.92] ±[0.11, 0.17, 0.11]

while the Fisher information for the center correction alone $\bar{F}_{bb}$ has the value

$$\bar{F}_{bb} = \begin{bmatrix} 7.686 & -0.325 & -0.630 \\ -0.325 & 0.013 & 0.027 \\ -0.630 & 0.027 & 0.052 \end{bmatrix} (\text{mG})^{-2} \tag{50b}$$

The information for all three axes is smaller for the center correction than for the centered estimate by a factor of from 25 to 2000. Hence, it is not expected that the center correction will be statistically significant, as is borne out by Table 1, where we observe the lack of change in the confidence intervals and the goodness of the center approximation. One the basis of this comparison of the two Fisher information matrices $\tilde{F}_{bb}$ and $\bar{F}_{bb}$ we would determine that the computation should be terminated after the evaluation of $\tilde{\mathbf{b}}^*$.

Consider Table 2 on the other hand. In this case the spacecraft is inertially stabilized and the observability of the magnetometer bias vector is expected to be much poorer. Thus we find for this case that

$$\tilde{F}_{bb} = \begin{bmatrix} 76.23 & -7.30 & -7.86 \\ -7.30 & 45.66 & 14.25 \\ -7.86 & 14.25 & 5.89 \end{bmatrix} (\text{mG})^{-2} \tag{51a}$$

while the Fisher information for the center correction alone $\bar{F}_{bb}$ has the value

$$\bar{F}_{bb} = \begin{bmatrix} 4.43 & 10.84 & 28.12 \\ 10.84 & 26.58 & 68.92 \\ 28.12 & 68.92 & 178.70 \end{bmatrix} (\text{mG})^{-2} \tag{51b}$$

From a comparison of the Fisher information matrix, we anticipate: an insignificant improvement in the estimate of b_1, because the (1, 1)-element of the Fisher information matrix for the center correction is smaller than the corresponding element for the center estimate by a factor of 17; a modest improvement in the estimate of b_2 because the (2, 2) elements of the two Fisher information matrices are comparable (the standard deviation of the error in the final estimate of b_2 should therefore be smaller than that for $\tilde{b}_2$ only by 20%); and a very noticeable improvement in the accuracy of b_3 by taking account of the center correction, because roughly 30 times more information resides in the center correction for that component than in the centered estimate.[6] This is borne out by Table 2, although there is noticeable improvement in all three components of the bias vector due to correlations. The correction of the third component, however, is the largest. In this case we must absolutely consider the center correction, since for one of the components of $\mathbf{b}^{*\prime}$ it provides most of the information. Since the centered estimation already provides a consistent estimate, it is likely (though not certain) that only a single iteration of the center correction will be necessary, even though the cost function is quartic.

Note in Tables 1 and 2 the similarity of the fractional parts of the estimates for the small and large values of the bias, the result of using the same seed for the simulation of random noise in each case. A different seed was used in Tables 1 and 2, however.

[6]This statement can only be very approximate because of the large correlations which are sometimes present in the Fisher information matrices.

Robustness of TWOSTEP

Thus far, both the estimator and the data have used identical statistical assumptions, in particular, it has been assumed that the fundamental magnetometer measurement noise is white and Gaussian. In general, it is neither of these, although estimators nearly always assume such a measurement noise model. This is the case for TWOSTEP. To test the sensitivity to these sweeping and not totally correct modeling assumptions, we have examined two cases. In the first case, we have replaced the white Gaussian noise sequence $\boldsymbol{\varepsilon}_k$ by a colored noise sequence described by a first-order Markov process driven by white noise. The "time constant" of the Markov process has been chosen to correspond to an orbital arc length of 18 deg, consistent with the correlation length associated with the neglected orders of the harmonic expansion of the magnetic field model. The power spectral density of the white-noise driving term has been chosen so that the covariance matrix of the stationary first-order Markov process will match that of the Gaussian white-noise model used in Tables 1 and 2. The results are shown in Table 3. The iteration index "1" is the centering approximation, further indices refer to iterations of the center correction. The quality of the estimates has deteriorated somewhat because the estimator now contains model errors. As a result, the actual errors are outside the error bounds computed by TWOSTEP based on its now incorrect assumptions on the nature of the measurement noise. However, the results are still quite good.

In a further numerical experiment, we have attempted to model the measurement noise as realistically as possible. To this end we have considered the properties of magnetometers constructed at NASA Goddard Space Flight Center [4]. These are characterized by a white noise and ripple effects of about $\sigma_o = 0.6$ mG per axis. In addition, the usable range of the magnetometer, from -600 mG to $+600$ mG is

TABLE 3. Performance of TWOSTEP for Colored Noise

Iteration	Bias Estimate (mG)
Spinning Spacecraft − magnetometer bias = [10., 20., 30.] mG	
1	[10.24, 20.68, 30.69] ± [0.09, 0.10, 0.10]
2	[10.18, 20.68, 30.69] ± [0.09, 0.10, 0.10]
Spinning Spacecraft − magnetometer bias = [100., 200., 300.] mG	
1	[100.91, 200.09, 300.16] ± [0.09, 0.10, 0.10]
2	[100.98, 200.09, 300.16] ± [0.09, 0.10, 0.10]
Inertially Stabilized Spacecraft − magnetometer bias = [10., 20., 30.] mG	
1	[10.80, 19.76, 32.91] ± [0.17, 0.26, 0.77]
2	[10.56, 20.16, 31.68] ± [0.09, 0.09, 0.11]
3	[10.56, 20.16, 31.69] ± [0.09, 0.09, 0.11]
Inertially Stabilized Spacecraft − magnetometer bias = [100., 200., 300.] mG	
1	[100.16, 198.71, 302.99] ± [0.17, 0.26, 0.76]
2	[99.67, 199.53, 300.47] ± [0.09, 0.09, 0.11]
3	[99.67, 199.52, 300.49] ± [0.09, 0.09, 0.11]

usually represented digitally by 12 bits, corresponding to a resolution of 0.29 mG $\equiv \Delta$. Thus, we may regard the telemetered field to be given (in counts) by

$$\mathbf{B}_k^{T/M} = \text{Int}[(A_k\mathbf{H}_k + \mathbf{b} + \mathbf{w}_k)/\Delta] \tag{52}$$

where Int(·) is the function which computes the greatest integer for each component of its argument, and $\mathbf{w}_k$ is Gaussian white noise whose covariance is given by $(0.6 \text{ mG})^2 I_{3\times3}$. The measurements would then be reconstructed from telemetry according to the prescription

$$\mathbf{B}_k = \Delta[\mathbf{B}_k^{T/M} + [0.5, 0.5, 0.5]^T] \tag{53}$$

For the model geomagnetic field model errors we have used the harmonic expansion coefficients of IGRF(85) up to order 10 to compute the raw measurements, but have used the coefficients only up to order 8 in the estimator. The TWOSTEP estimator assumes only the known random and quantization errors, that is, it assumes

$$\Sigma_k = \left(\sigma_o^2 + \frac{\Delta^2}{12}\right) I_{3\times3} \tag{54}$$

The results of the magnetometer bias determination given this mismatch between measurement noise and estimator are shown in Table 4. The results again clearly show errors that are significantly larger than the statistical limits computed from the estimators error model but are quite acceptable also in this case. Note that proportionately the agreement is greater for the larger biases in both Table 3 and Table 4, because the modeling errors are proportionately smaller. We see in these examples of mismodeling some of the few cases where more than one iteration of the center correction has been needed. The result of that further iteration can hardly be called significant, however.

TABLE 4. Performance of TWOSTEP for "Realistic" Measurement Noise Simulation

Iteration	Bias Estimate (mG)
Spinning Spacecraft − magnetometer bias = [10., 20., 30.] mG	
1	[9.82, 20.14, 30.06] ± [0.03, 0.03, 0.03]
2	[9.77, 20.15, 30.06] ± [0.03, 0.03, 0.03]
Spinning Spacecraft − magnetometer bias = [100., 200., 300.] mG	
1	[99.77, 200.13, 300.00] ± [0.03, 0.03, 0.03]
2	[99.74, 200.12, 300.00] ± [0.03, 0.03, 0.03]
Inertially Stabilized Spacecraft − magnetometer bias = [10., 20., 30.] mG	
1	[9.85, 20.26, 30.53] ± [0.05, 0.08, 0.23]
2	[9.73, 20.45, 30.52] ± [0.03, 0.03, 0.03]
Inertially Stabilized Spacecraft − magnetometer bias = [100., 200., 300.] mG	
1	[98.82, 200.36, 300.02] ± [0.05, 0.08, 0.23]
2	[99.82, 200.36, 300.02] ± [0.03, 0.03, 0.03]

Discussion

A new algorithm, TWOSTEP, has been developed, which is efficient and robust, and which leads to a consistent estimate of the magnetometer bias at both steps of the algorithm. Its ability to converge in all cases (nearly 200 have been simulated by the authors) is due to the fact that, if the magnetometer bias is observable at all, the centering approximation will yield a consistent and unambiguous result. Thus, the center correction, in most cases, makes little improvement in the estimate.

An important component in the development of the algorithm was the correct treatment of the correlations introduced by the centering process. We have shown that the correct treatment leads to a the centered negative-log-likelihood function which is the sum of squares.

An obvious characteristic of the centered estimate, the first step in TWOSTEP is that it is often good enough.[7] The Fisher information associated with $\tilde{\mathbf{b}}$ genuinely characterizes the quality of the centered estimate. A comparison of this and the Fisher information associated with the center term can be used to decide whether it is worthwhile to carry out the center correction. This was demonstrated explicitly in the previous section. We see that a careful statistical treatment of the magnetometer bias gives us many more insights into the behavior of the estimator.

Note that the variances σ_k^2 given in equation (7b) are functions of $\mathbf{b}$. We have taken them to be functions of the true value of $\mathbf{b}$ and not of the corresponding model variable which appears in the cost function. Had we taken $\mathbf{b}$ to be a parameter of σ_k^2 also, then we would have differentiated also the factors $1/\sigma_k^2$ and the terms $\log \sigma_k^2$ appearing in equation (10). This latter approach would, in principal, have been more correct, but might have led to convergence problems because of the nonlinearity. However, a consistent estimate of $\mathbf{b}$ can be obtained for any set of the values for the σ_k^2, so that the added complication of making σ_k^2 a function of $\mathbf{b}$ in the cost function is not justified. Nonetheless, for consistency, once $\tilde{\mathbf{b}}^*$ has been determined from our initial set of σ_k^2, which were computed using $\mathbf{b} = \mathbf{0}$, we have recomputed the σ_k^2 using $\tilde{\mathbf{b}}^*$ as the "true" value and repeated the centering step to obtain an "improved" but hardly very different value for $\tilde{\mathbf{b}}^*$. This, our two-step method typically incorporates at least two iterations in the first step alone, and combines both scoring and fixed-point techniques. In a more realistic calculation, of course, one should give up the approximation that the effective magnetometer errors are isotropic and white. However, experience has shown us that the estimates are not very sensitive to the choice of the σ_k^2, at least not for the orbit considered, which never comes within 50 degrees of the poles. Thus, the choice of the σ_k^2 does not seem to be important to the estimation problem. The difficulties that have been encountered up to now in estimating the magnetometer bias vector without knowledge of the attitude did not arise from an unrealistic modeling of the error levels but rather from the improper treatment of the non-quadratic nature of the cost function. Our goal in developing the TWOSTEP algorithm was not to make insignificant gains in computation times but to develop an algorithm which was more reliable than its predecessors.

More interesting would be the computation of the parameters of Σ_k, which are of fundamental importance. However, experience has shown that the most significant errors are those associated with the magnetic field model, which, to be meaningful, should be represented in a topocentric coordinate system associated with the geo-

[7]Perhaps we should call it ONESTEP, not very danceable until one recalls the "hop!"

magnetic dipole field. Such a representation of Σ_k is impossible without a knowledge of the spacecraft attitude. Therefore, the estimation of error-level parameters, except at the crudest level, is not appropriate to the present study. For a detailed discussion of the errors in geomagnetic field models the reader is referred to Langel [8] and the two special issues devoted to the Magsat mission [9, 10].

TWOSTEP has been shown to work well even when the assumption of white Gaussian statistics is incorrect. The main reason for this is that the separation of the cost function into $\tilde{J}(\mathbf{b})$ and $\bar{J}(\mathbf{b})$ doesn't really depend on the statistical assumptions.[8] Thus, the TWOSTEP algorithm will lead to an exact minimization of the cost function even if the statistical assumptions are not justified. There is still a price to be paid for incorrectly modeled statistics, however, which is that the computed confidence intervals will not be correct, as we have already seen in the numerical examples.

In summary: we have developed a new algorithm for magnetometer bias determination in the absence of attitude information which is efficient and robust. The algorithm begins by centering the data and computes an estimate of the magnetometer bias $\tilde{\mathbf{b}}^*$ from this centered data in one step while treating the correlations between centered measurements correctly. Assuming the magnetometer errors (apart from the bias) to be white and Gaussian, the estimate so generated will be statistically consistent. The Fisher information matrix of the centered data provides a characterization of the accuracy of the estimate, and the Fisher information matrix of the lone center measurement (evaluated at $\tilde{\mathbf{b}}^*$) provides a direct assessment of whether significant improvement can be obtained for the magnetometer bias estimate by taking the discarded center measurement into account. If it is important to take the center measurement into account, then a complete estimate using all data can be obtained by considering only the centered estimate, the associated Fisher information matrix, and the center measurement (and the center variance $\bar{\sigma}^2$). The centered data need not be reprocessed. Since the centered estimate already provides a good value for the bias, convergence of the center correction is rapid.

TWOSTEP is certainly more sophisticated statistically and more capable than its predecessor algorithms for attitude-independent magnetometer calibration, more efficient computationally, and more reliable. Comparisons of different algorithms are the subject of [5]. Perhaps, most importantly, the new algorithm makes manifest the physical quantities which determine the behavior of the bias estimator. We hasten to point out that the algorithm can only be as good as the validity of its statistical model. If the effective measurement noise is incorrectly modeled, then the new algorithm will certainly show systematic or at least larger errors. This has been seen in some of the cases examined above where the measurement noise has been intentionally mismodeled. Although the errors levels were much larger than the naive statistical predictions in this case, as expected, the accuracy level was certainly usable.

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[8]Of course, if the assumption of Gaussian statistics does not hold then $\tilde{J}(\mathbf{b})$ and $\bar{J}(\mathbf{b})$ will not be statistically independent.

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Appendix: the Centered Negative-Log-Likelihood Function

To calculate det $\widetilde{\mathcal{R}}$, consider the probability density of the measurements given the parameters $p(\{z_k, k = 1, \ldots, N\} \,|\, \mathbf{b})$. We write this as a conditional probability on the value of $\mathbf{b}$. From the definition of the conditional probability, we can write this also as

$$\begin{aligned} p_{z_1,\ldots,z_N|\mathbf{b}}(z'_1, \ldots, z'_N \,|\, \mathbf{b}) &= p_{\tilde{z}_1,\ldots,\tilde{z}_{N-1},\bar{z}|\mathbf{b}}(\tilde{z}'_1, \ldots, \tilde{z}'_{N-1}, \bar{z}' \,|\, \mathbf{b}) \\ &= p_{\tilde{z}_1,\ldots,\tilde{z}_{N-1},\bar{z}|\mathbf{b}}(\tilde{z}'_1, \ldots, \tilde{z}'_{N-1} \,|\, \bar{z}', \mathbf{b}) p_{\bar{z}|\mathbf{b}}(\bar{z}' \,|\, \mathbf{b}) \\ &= p_{\tilde{z}_1,\ldots,\tilde{z}_{N-1},\bar{z}|\mathbf{b}}(\tilde{z}'_1, \ldots, \tilde{z}'_{N-1} \,|\, \mathbf{b}) p_{\bar{z}|\mathbf{b}}(\bar{z}' \,|\, \mathbf{b}) \end{aligned} \tag{A1}$$

The last step results from the fact that $\bar{z}$ is uncorrelated with $z_1, \ldots, z_{N-1}$. Considered as a function of the parameters, the probability density function is called the likelihood function. Taking the negative logarithm of both members of equation (A1) we obtain an equivalent relation (in even more defective notation) for the corresponding negative-log-likelihood functions

$$J(\mathbf{b} \,|\, z'_1, \ldots, z'_N) = J(\mathbf{b} \,|\, \tilde{z}'_1, \ldots, \tilde{z}'_{N-1}) + J(\mathbf{b} \,|\, \bar{z}') \tag{A2}$$

or, in our earlier notation,

$$J(\mathbf{b}) = \tilde{J}(\mathbf{b}) + \bar{J}(\mathbf{b}) \tag{A3}$$

If the explicit expressions for $J(\mathbf{b})$ and $\bar{J}(\mathbf{b})$ from equations (10) and (43), respectively, are now substituted and equation (A3) is solved for $\tilde{J}(\mathbf{b})$, we obtain

$$\begin{aligned} \tilde{J}(\mathbf{b}) &= \frac{1}{2} \sum_{k=1}^{N} \frac{1}{\sigma_k^2} (\tilde{z}'_k - 2\tilde{\mathbf{B}}_k \cdot \mathbf{b} - \tilde{\mu}_k)^2 \\ &\quad + \frac{1}{2} \left(\sum_{k=1}^{N} \log \sigma_k^2 - \log \bar{\sigma}^2 \right) + \frac{1}{2} (N - 1) \log 2\pi \end{aligned} \tag{A4}$$

The first term is just the bias-dependent part of $\tilde{J}(\mathbf{b})$ as given in equation (32). We now have in addition that

$$\log \det \widetilde{\mathcal{R}} = \sum_{k=1}^{N} \log \sigma_k^2 - \log \bar{\sigma}^2 \tag{A5}$$

or

$$\det \widetilde{\mathcal{R}} = \frac{\sigma_1^2 \sigma_2^2 \cdots \sigma_N^2}{\bar{\sigma}^2} \tag{A6}$$

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