A New Angle on the Euler Angles

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Abstract

We present a generalization of the Euler angles to axes beyond the twelve conventional sets. The generalized Euler axes must satisfy the constraint that the first and the third are orthogonal to the second; but the angle between the first and third is arbitrary, rather than being restricted to the values 0 and $\pi/2$, as in the conventional sets. This is the broadest generalization of the Euler angles that provides a representation of an arbitrary rotation matrix. The kinematics of the generalized Euler angles and their relation to the attitude matrix are presented. As a side benefit, the equations for the generalized Euler angles are universal in that they incorporate the equations for the twelve conventional sets of Euler angles in a natural way.

Introduction

It is well known that a rotation can be represented by a single rotation about a single axis, where the rotation axis is allowed to vary according to the rotation [1–7]. It is often more convenient to represent a general rotation as the product of three successive rotations about axes whose orientations are specified *a priori*. These parameterizations of rotations, well known as the Euler angle parameterizations [1–7], can be written

$$R(\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2, \hat{\mathbf{n}}_3; \varphi, \vartheta, \psi) \equiv R(\hat{\mathbf{n}}_3, \psi) R(\hat{\mathbf{n}}_2, \vartheta) R(\hat{\mathbf{n}}_1, \varphi), \tag{1}$$

where the carets denote unit vectors, and $R(\hat{\mathbf{n}}, \varphi)$ represents a rotation by angle φ about axis $\hat{\mathbf{n}}$. For the conventional Euler angles, the rotation axes are selected from the set $\{\hat{\mathbf{1}}, \hat{\mathbf{2}}, \hat{\mathbf{3}}\}$ where

$$\hat{\mathbf{1}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \hat{\mathbf{2}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \hat{\mathbf{3}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{2}$$

The conventional Euler rotations are generally designated by the three indices, for example

$$R_{213}(\varphi, \vartheta, \psi) \equiv R(\hat{\mathbf{2}}, \hat{\mathbf{1}}, \hat{\mathbf{3}}; \varphi, \vartheta, \psi) = R(\hat{\mathbf{3}}, \psi) R(\hat{\mathbf{1}}, \vartheta) R(\hat{\mathbf{2}}, \varphi). \tag{3}$$

If the Euler sequence is to represent a general rotation matrix, two successive rotations cannot be about the same axis, which is to say that $\hat{\mathbf{n}}_1 \neq \hat{\mathbf{n}}_2$ and $\hat{\mathbf{n}}_2 \neq \hat{\mathbf{n}}_3$. This leaves twelve possible sets of conventional Euler axes: six symmetric sets designated 121, 232, 313, 131, 212, and 323, and six asymmetric sets designated 123, 231, 312, 132, 213, and 321.

We show in the present work that the Euler angles can be extended to a much larger set. The generalized Euler axes can be any three unit vectors such that both the first and the third are orthogonal to the second. The angle between the first and third axes is arbitrary, rather than being restricted to the values 0 and $\pi/2$ as is the case for the conventional sets. We show that this is a necessary and sufficient condition for the generalized Euler angles to provide a universal representation of rotation matrices. We derive expressions for the generalized Euler angles in terms of the rotation matrix and kinematic equations for these angles, and discuss the 'gimbal-lock' singularity of this parameterization.

Necessary Conditions for the Generalized Euler Angles

For the generalized Euler angles to represent a general rotation, it is necessary and sufficient that the rotation matrix of equation (1) be capable of mapping any unit vector $\hat{\mathbf{u}}$ into any other unit vector $\hat{\mathbf{v}}$. That is, there must exist angles φ, ϑ, and ψ such that the equation

$$\hat{\mathbf{v}} = R(\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2, \hat{\mathbf{n}}_3; \varphi, \vartheta, \psi)\hat{\mathbf{u}} \tag{4}$$

has a solution for given $\hat{\mathbf{n}}_1$, $\hat{\mathbf{n}}_2$, $\hat{\mathbf{n}}_3$, $\hat{\mathbf{u}}$, and $\hat{\mathbf{v}}$. In order to show the necessity of the conditions on the rotation axes, we can take $\hat{\mathbf{u}}$ equal to $\hat{\mathbf{n}}_1$ and only look at the component of this equation along $\hat{\mathbf{n}}_3$. That is, it is certainly necessary that

$$\hat{\mathbf{n}}_3 \bullet \hat{\mathbf{v}} = \hat{\mathbf{n}}_3^T R(\hat{\mathbf{n}}_3, \psi) R(\hat{\mathbf{n}}_2, \vartheta) R(\hat{\mathbf{n}}_1, \varphi)\hat{\mathbf{n}}_1 = \hat{\mathbf{n}}_3^T R(\hat{\mathbf{n}}_2, \vartheta)\hat{\mathbf{n}}_1, \tag{5}$$

where we have used equation (1) and recalled that the axis of rotation is invariant under a rotation. Inserting the explicit form of the rotation matrix [5]

$$R(\hat{\mathbf{n}}, \zeta) = \cos\zeta\, I_{3\times 3} - \sin\zeta[\hat{\mathbf{n}}\times] + (1-\cos\zeta)\hat{\mathbf{n}}\hat{\mathbf{n}}^T, \tag{6}$$

with

$$[\hat{\mathbf{n}}\times] \equiv \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix}, \tag{7}$$

gives the necessary condition

$$\begin{aligned} \hat{\mathbf{n}}_3 \cdot \hat{\mathbf{v}} &= \hat{\mathbf{n}}_3 \cdot [\cos\vartheta\, \hat{\mathbf{n}}_1 - \sin\vartheta(\hat{\mathbf{n}}_2 \times \hat{\mathbf{n}}_1) + (1-\cos\vartheta)\hat{\mathbf{n}}_2(\hat{\mathbf{n}}_2 \cdot \hat{\mathbf{n}}_1)] \\ &= (\hat{\mathbf{n}}_2 \cdot \hat{\mathbf{n}}_3)(\hat{\mathbf{n}}_2 \cdot \hat{\mathbf{n}}_1) + \sin\vartheta[\hat{\mathbf{n}}_3 \cdot (\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2)] + \cos\vartheta(\hat{\mathbf{n}}_2 \times \hat{\mathbf{n}}_3)\cdot(\hat{\mathbf{n}}_2 \times \hat{\mathbf{n}}_1). \end{aligned} \tag{8}$$

Now let

$$\beta \equiv (\hat{\mathbf{n}}_2 \cdot \hat{\mathbf{n}}_3)(\hat{\mathbf{n}}_2 \cdot \hat{\mathbf{n}}_1) \tag{9}$$

and B be the positive square root of

$$
\begin{aligned}
B^2 &\equiv \left[\hat{\mathbf{n}}_3 \cdot (\hat{\mathbf{n}}_2 \times \hat{\mathbf{n}}_1)\right]^2 + \left[(\hat{\mathbf{n}}_2 \times \hat{\mathbf{n}}_3) \cdot (\hat{\mathbf{n}}_2 \times \hat{\mathbf{n}}_1)\right]^2 \\
&= \det\left(\left[\hat{\mathbf{n}}_1 \vdots \hat{\mathbf{n}}_2 \vdots \hat{\mathbf{n}}_3\right]^T \left[\hat{\mathbf{n}}_1 \vdots \hat{\mathbf{n}}_2 \vdots \hat{\mathbf{n}}_3\right]\right) + \left[(\hat{\mathbf{n}}_2 \times \hat{\mathbf{n}}_3) \cdot (\hat{\mathbf{n}}_2 \times \hat{\mathbf{n}}_1)\right]^2 \\
&= \det\begin{bmatrix} 1 & \hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_2 & \hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_3 \\ \hat{\mathbf{n}}_2 \cdot \hat{\mathbf{n}}_1 & 1 & \hat{\mathbf{n}}_2 \cdot \hat{\mathbf{n}}_3 \\ \hat{\mathbf{n}}_3 \cdot \hat{\mathbf{n}}_1 & \hat{\mathbf{n}}_3 \cdot \hat{\mathbf{n}}_2 & 1 \end{bmatrix} + \left[\hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_3 - (\hat{\mathbf{n}}_2 \cdot \hat{\mathbf{n}}_3)(\hat{\mathbf{n}}_2 \cdot \hat{\mathbf{n}}_1)\right]^2 = \left|\hat{\mathbf{n}}_2 \times \hat{\mathbf{n}}_3\right|^2 \left|\hat{\mathbf{n}}_2 \times \hat{\mathbf{n}}_1\right|^2.
\end{aligned}
\tag{10}
$$

It is clear from the final expression that $B \le 1$. Now equation (8) can be written as

$$\hat{\mathbf{n}}_3 \cdot \hat{\mathbf{v}} = \beta + B\cos(\vartheta - \lambda), \tag{11}$$

where

$$\lambda \equiv \mathrm{ATAN2}\left[\hat{\mathbf{n}}_3 \cdot (\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2),\ (\hat{\mathbf{n}}_2 \times \hat{\mathbf{n}}_3) \cdot (\hat{\mathbf{n}}_2 \times \hat{\mathbf{n}}_1)\right]. \tag{12}$$

As ϑ varies over its range, the right side of equation (12) takes only values between $\beta - B$ and $\beta + B$, so a solution will exist for ϑ only if

$$\beta - B \le \hat{\mathbf{n}}_3 \cdot \hat{\mathbf{v}} \le \beta + B. \tag{13}$$

However, $\hat{\mathbf{n}}_3 \cdot \hat{\mathbf{v}}$ can assume any value between – 1 and + 1, so it is clear from equations (9) and (10) that the coefficients β and B must have the values

$$B = 1 \qquad \text{and} \qquad \beta = 0 \tag{14}$$

This means that

$$\hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_2 = 0 \qquad \text{and} \qquad \hat{\mathbf{n}}_2 \cdot \hat{\mathbf{n}}_3 = 0 \tag{15}$$

or equivalently that $\hat{\mathbf{n}}_2$ be perpendicular to both $\hat{\mathbf{n}}_1$ and $\hat{\mathbf{n}}_3$. With this restriction, equation (12) simplifies to

$$\lambda \equiv \mathrm{ATAN2}\left[\hat{\mathbf{n}}_3 \cdot (\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2),\ \hat{\mathbf{n}}_3 \cdot \hat{\mathbf{n}}_1\right] \tag{16}$$

and then

$$\hat{\mathbf{n}}_3 = \cos\lambda\, \hat{\mathbf{n}}_1 + \sin\lambda (\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2) = R(\hat{\mathbf{n}}_2, \lambda)\hat{\mathbf{n}}_1. \tag{17}$$

Thus λ is the angle of the rotation about $\hat{\mathbf{n}}_2$ that takes $\hat{\mathbf{n}}_1$ into $\hat{\mathbf{n}}_3$.

Sufficiency of the Generalized Euler Angle Parameterization

The rotation matrix can be written as the product

$$
\begin{aligned}
R(\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2, \hat{\mathbf{n}}_3; \varphi, \vartheta, \psi) &\equiv R(\hat{\mathbf{n}}_3, \psi) R(\hat{\mathbf{n}}_2, \vartheta) R(\hat{\mathbf{n}}_1, \varphi) = R\big(R(\hat{\mathbf{n}}_2, \lambda)\hat{\mathbf{n}}_1, \psi\big) R(\hat{\mathbf{n}}_2, \vartheta) R(\hat{\mathbf{n}}_1, \varphi) \\
&= R(\hat{\mathbf{n}}_2, \lambda) R(\hat{\mathbf{n}}_1, \psi) R^T(\hat{\mathbf{n}}_2, \lambda) R(\hat{\mathbf{n}}_2, \vartheta) R(\hat{\mathbf{n}}_1, \varphi) \\
&= R(\hat{\mathbf{n}}_2, \lambda) R(\hat{\mathbf{n}}_1, \psi) R(\hat{\mathbf{n}}_2, \vartheta - \lambda) R(\hat{\mathbf{n}}_1, \varphi) = R(\hat{\mathbf{n}}_2, \lambda) R(\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2, \hat{\mathbf{n}}_1; \varphi, \vartheta - \lambda, \psi).
\end{aligned}
\tag{18}
$$

If this is to represent an arbitrary proper orthogonal matrix A, we must be able to find angles φ, ϑ, and ψ such that

$$A = R(\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2, \hat{\mathbf{n}}_3; \varphi, \vartheta, \psi) = R(\hat{\mathbf{n}}_2, \lambda) R(\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2, \hat{\mathbf{n}}_1; \varphi, \vartheta - \lambda, \psi), \tag{19}$$

or, equivalently, to find angles φ, $\vartheta' \equiv \vartheta - \lambda$, and ψ such that

$$R(\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2, \hat{\mathbf{n}}_1; \varphi, \vartheta', \psi) = R^T(\hat{\mathbf{n}}_2, \lambda) A. \tag{20}$$

The matrix on the right side of this equation ranges over the group of proper orthogonal matrices as A ranges over this group. Thus our generalized Euler sequence can represent an arbitrary rotation if the matrix on the left side of equation (20) can represent an arbitrary rotation. To establish this fact, it is sufficient to show that this matrix can take the vectors in some orthonormal basis into an arbitrary orthonormal triad. We will take this basis to be $\{\hat{\mathbf{n}}_1,\ \hat{\mathbf{n}}_2,\ \hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2\}$. Thus we must be able to find angles φ, ϑ', and ψ such that

$$R(\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2, \hat{\mathbf{n}}_1; \varphi, \vartheta', \psi)\hat{\mathbf{n}}_1 = \hat{\mathbf{v}}_1, \tag{21}$$

where $\hat{\mathbf{v}}_1$ is an arbitrary unit vector, and

$$R(\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2, \hat{\mathbf{n}}_1; \varphi, \vartheta', \psi)\hat{\mathbf{n}}_2 = \hat{\mathbf{v}}_2, \tag{22}$$

where $\hat{\mathbf{v}}_2$ is a unit vector in the plane perpendicular to $\hat{\mathbf{v}}_1$, but is otherwise arbitrary. Then the proper orthogonality of $R(\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2, \hat{\mathbf{n}}_1; \varphi, \vartheta', \psi)$ ensures that it will map $\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2$ into $\hat{\mathbf{v}}_1 \times \hat{\mathbf{v}}_2$.

Equation (21) can be written, using equations (1) and (6), as

$$\begin{aligned} \hat{\mathbf{v}}_1 &= R(\hat{\mathbf{n}}_1, \psi)\left[\cos\vartheta'\,\hat{\mathbf{n}}_1 - \sin\vartheta'(\hat{\mathbf{n}}_2 \times \hat{\mathbf{n}}_1)\right] \\ &= \cos\vartheta'\,\hat{\mathbf{n}}_1 + \sin\vartheta' \sin\psi\,\hat{\mathbf{n}}_2 + \sin\vartheta' \cos\psi(\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2). \end{aligned} \tag{23}$$

Since $\{\hat{\mathbf{n}}_1,\ \hat{\mathbf{n}}_2,\ \hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2\}$ is a basis, it is clear that ϑ' and ψ can be chosen so that $\hat{\mathbf{v}}_1$ is an arbitrary vector. Equation (22) gives

$$\hat{\mathbf{v}}_2 = R(\hat{\mathbf{n}}_1, \psi) R(\hat{\mathbf{n}}_2, \vartheta')\left[\cos\varphi\,\hat{\mathbf{n}}_2 - \sin\varphi(\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2)\right] = \cos\varphi\,\hat{\mathbf{u}}_1 + \sin\varphi\,\hat{\mathbf{u}}_2, \tag{24}$$

where

$$\hat{\mathbf{u}}_1 \equiv R(\hat{\mathbf{n}}_1, \psi) R(\hat{\mathbf{n}}_2, \vartheta')\hat{\mathbf{n}}_2 = R(\hat{\mathbf{n}}_1, \psi)\hat{\mathbf{n}}_2 = \cos\psi\,\hat{\mathbf{n}}_2 - \sin\psi(\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2), \tag{25}$$

and

$$\begin{aligned} \hat{\mathbf{u}}_2 &\equiv -R(\hat{\mathbf{n}}_1, \psi) R(\hat{\mathbf{n}}_2, \vartheta')\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2 = -R(\hat{\mathbf{n}}_1, \psi)\left[\cos\vartheta'(\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2) - \sin\vartheta'\,\hat{\mathbf{n}}_1\right] \\ &= \sin\vartheta'\,\hat{\mathbf{n}}_1 - \cos\vartheta' \sin\psi\,\hat{\mathbf{n}}_2 - \cos\vartheta' \cos\psi(\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2). \end{aligned} \tag{26}$$

It is clear from equations (23), (25), and (26) that $\hat{\mathbf{u}}_1$ and $\hat{\mathbf{u}}_2$ form an orthogonal basis for the plane perpendicular to $\hat{\mathbf{v}}_1$. Thus equation (24) shows that φ can be chosen such that $\hat{\mathbf{v}}_2$ is any vector in this plane.

This completes the demonstration that the generalized Euler angles, subject to the constraint of equation (15), can represent an arbitrary rotation. Since the conventional angles are a subset of the generalized Euler angles, it incidentally provides an explicit proof that the conventional Euler angles are similarly general.

Relation to the Conventional Euler Angles

Each of the conventional Euler angle sets is a subset of the class of generalized Euler angles, characterized by a specific choice of axes and a corresponding value of the angle λ. It is easily seen from equation (12) or (17) that the symmetric sets of axes (121, 232, 313, 131, 212, and 323) have $\lambda = 0$, the even permutation asymmetric sets (123, 231, and 312) have $\lambda = \pi/2$, and the odd permutation asymmetric sets (132, 213, and 321) have $\lambda = -\pi/2$. With these substitutions, all the equations derived in this paper are applicable to the conventional Euler angles. Thus the results of this paper include universal formulas applicable to all Euler angles, conventional or generalized.

Extraction of the Generalized Euler Angles

The rotation matrix is simply defined in terms of the generalized Euler angles by equation (1). We now turn to the converse problem, the extraction of the generalized Euler angles from a rotation matrix. Equation (11), with the constraint of equation (14), gives

$$\hat{\mathbf{n}}_3 \cdot \hat{\mathbf{v}} = \cos(\vartheta - \lambda). \tag{27}$$

We recall from equations (4) and (5) that $\hat{\mathbf{v}} = A\hat{\mathbf{n}}_1$, where A is the rotation matrix that is being parameterized, so this equation can be solved for ϑ, yielding

$$\vartheta = \lambda \pm \mathrm{ACOS}\left(\hat{\mathbf{n}}_3^T A \hat{\mathbf{n}}_1\right), \tag{28}$$

where ACOS denotes the principal value of the inverse cosine function, which returns a value between 0 and π. The twofold sign ambiguity in equation (28) is present in the conventional Euler angle representations as well, but it is usually avoided by restricting ϑ to the range $0 \le \vartheta \le \pi$ for the symmetric sets of axes or $-\pi/2 \le \vartheta \le \pi/2$ for the asymmetric sets. A similar resolution of the ambiguity for the generalized Euler angle case would be to take the sign of the second term in equation (28) to be positive for $\lambda \le 0$ and negative for $\lambda > 0$. This would ensure that the values of ϑ for any particular choice of axes would always be in some interval of length π of the range $-\pi < \vartheta \le \pi$. We will not assume that this convention has been adopted, however.

Equation (28) is analogous to the procedure for finding the second Euler angle in one of the conventional sets in terms of one of the elements of the rotation matrix. The other angles are expressed in terms of the other two elements of the same row or of the same column. This motivates us to consider the four quantities

$$\hat{\mathbf{n}}_2^T A \hat{\mathbf{n}}_1 = \left[\cos\psi\, \hat{\mathbf{n}}_2 - \sin\psi(\hat{\mathbf{n}}_2 \times \hat{\mathbf{n}}_3)\right]^T \left[\cos\vartheta\, \hat{\mathbf{n}}_1 + \sin\vartheta(\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2)\right] = \sin\psi \sin(\vartheta - \lambda), \tag{29}$$

$$(\hat{\mathbf{n}}_2 \times \hat{\mathbf{n}}_3)^T A \hat{\mathbf{n}}_1 = \left[\cos\psi (\hat{\mathbf{n}}_2 \times \hat{\mathbf{n}}_3) + \sin\psi\, \hat{\mathbf{n}}_2\right]^T \left[\cos\vartheta\, \hat{\mathbf{n}}_1 + \sin\vartheta (\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2)\right] \\ = -\cos\psi \sin(\vartheta - \lambda), \tag{30}$$

$$\hat{\mathbf{n}}_3^T A \hat{\mathbf{n}}_2 = \left[\cos\vartheta\, \hat{\mathbf{n}}_3 + \sin\vartheta (\hat{\mathbf{n}}_2 \times \hat{\mathbf{n}}_3)\right]^T \left[\cos\varphi\, \hat{\mathbf{n}}_2 - \sin\varphi (\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2)\right] = \sin\varphi \sin(\vartheta - \lambda), \tag{31}$$

and

$$\hat{\mathbf{n}}_3^T A (\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2) = \left[\cos\vartheta\, \hat{\mathbf{n}}_3 + \sin\vartheta (\hat{\mathbf{n}}_2 \times \hat{\mathbf{n}}_3)\right]^T \left[\cos\varphi (\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2) + \sin\varphi\, \hat{\mathbf{n}}_2\right] \\ = -\cos\varphi \sin(\vartheta - \lambda). \tag{32}$$

Define σ as the sign

$$\sigma \equiv \mathrm{sign}\left[\sin(\vartheta - \lambda)\right]. \tag{33}$$

This sign is not a variable, but is fixed for any set of generalized Euler axes. It is, in fact, the same as the sign of the second term on the right side of equation (28). With this definition, we can find the other two generalized Euler angles by

$$\varphi = \mathrm{ATAN2}\left[\sigma\, \hat{\mathbf{n}}_3^T A \hat{\mathbf{n}}_2,\; -\sigma\, \hat{\mathbf{n}}_3^T A (\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2)\right] \tag{34}$$

and

$$\psi = \mathrm{ATAN2}\left[\sigma\, \hat{\mathbf{n}}_2^T A \hat{\mathbf{n}}_1,\; -\sigma\, (\hat{\mathbf{n}}_2 \times \hat{\mathbf{n}}_3)^T A \hat{\mathbf{n}}_1\right]. \tag{35}$$

The equations for the conventional Euler angles are, of course, special cases of these equations.

Kinematics

The kinematic equations for the generalized Euler angles are straightforward generalizations of the corresponding expressions for the conventional Euler angles. The body-referenced angular velocity vector is given by

$$\omega = \dot{\psi}\, \hat{\mathbf{n}}_3 + \dot{\vartheta} R(\hat{\mathbf{n}}_3, \psi) \hat{\mathbf{n}}_2 + \dot{\varphi} R(\hat{\mathbf{n}}_3, \psi) R(\hat{\mathbf{n}}_2, \vartheta) \hat{\mathbf{n}}_1 = R(\hat{\mathbf{n}}_3, \psi) S \begin{bmatrix} \dot{\varphi} \\ \dot{\vartheta} \\ \dot{\psi} \end{bmatrix}, \tag{36}$$

where S is the 3×3 matrix

$$S \equiv \left[\hat{\mathbf{n}}' \vdots \hat{\mathbf{n}}_2 \vdots \hat{\mathbf{n}}_3\right] \tag{37}$$

with

$$\hat{\mathbf{n}}' \equiv R(\hat{\mathbf{n}}_2, \vartheta) \hat{\mathbf{n}}_1 = R(\hat{\mathbf{n}}_2, \vartheta - \lambda) \hat{\mathbf{n}}_3 = \cos(\vartheta - \lambda) \hat{\mathbf{n}}_3 - \sin(\vartheta - \lambda)(\hat{\mathbf{n}}_2 \times \hat{\mathbf{n}}_3). \tag{38}$$

The second step in equation (38) makes use of equation (17). The inverse of equation (37) gives the time derivatives of the Euler angles in terms of the angular velocity:

$$\begin{bmatrix} \dot{\varphi} \\ \dot{\vartheta} \\ \dot{\psi} \end{bmatrix} = S^{-1} R^T (\hat{\mathbf{n}}_3, \psi) \omega. \tag{39}$$

The determinant of the matrix S is given by

$$\det S = \hat{\mathbf{n}}' \cdot (\hat{\mathbf{n}}_2 \times \hat{\mathbf{n}}_3) = -\sin(\vartheta - \lambda), \tag{40}$$

and its inverse is

$$\begin{aligned} S^{-1} &= (\det S)^{-1} \left[\hat{\mathbf{n}}_2 \times \hat{\mathbf{n}}_3 \,\vdots\, \hat{\mathbf{n}}_3 \times \hat{\mathbf{n}}' \,\vdots\, \hat{\mathbf{n}}' \times \hat{\mathbf{n}}_2 \right]^T \\ &= \left[\sin(\vartheta - \lambda) \right]^{-1} \left[\hat{\mathbf{n}}_3 \times \hat{\mathbf{n}}_2 \,\vdots\, \sin(\vartheta - \lambda) \hat{\mathbf{n}}_2 \,\vdots\, \sin(\vartheta - \lambda) \hat{\mathbf{n}}_3 - \cos(\vartheta - \lambda)(\hat{\mathbf{n}}_3 \times \hat{\mathbf{n}}_2) \right]^T . \end{aligned} \tag{41}$$

The kinematic equations for the conventional Euler angles are special cases of these equations.

Singularity of the Parameterization

It is clear from equations (39) and (41) that the kinematic equations for φ and ψ are singular when $\sin(\vartheta - \lambda) = 0$. It is also clear that the kinematic equation for ϑ is not singular at this or any other point. The mathematical singularity reflects the fact that the axes $\hat{\mathbf{n}}_1$ and $\hat{\mathbf{n}}_3$ coincide when $\sin(\vartheta - \lambda) = 0$, so the rotations about these two axes are not independent. This situation is known as gimbal lock, since it is related to the serious problem occurring in gimballed inertial reference platforms for which the Euler angles are physical gimbal angles, and the required infinite rates cannot be attained by physical actuators. It is worth mentioning, however, that the numerical errors accumulated in integration of the kinematic equations through the gimbal-lock singularity can be surprisingly small in practice [8].

It is interesting to note that the combination $\cos(\vartheta - \lambda)\dot{\varphi} + \dot{\psi}$ is nonsingular in the limit that $\sin(\vartheta - \lambda) = 0$, so that this combination of these two angular rates continues to be significant. The formulas for extracting φ and ψ from the attitude matrix, equations (34) and (35), are both undefined in this limit, however. It is possible to extract information from other elements of the rotation matrix to give a correct value to the linear combination of φ and ψ that remains well-defined, and an explicit procedure to accomplish this has been worked out for the conventional Euler angles [9]. The generalization of this procedure to the generalized Euler angles is straightforward. With a moderate amount of effort, we can derive the following relationships between the 'matrix elements' of A and the generalized Euler angles:

$$\hat{\mathbf{n}}_2^T A \hat{\mathbf{n}}_2 = \cos\varphi \cos\psi - \sin\varphi \sin\psi \cos(\vartheta - \lambda), \tag{42}$$

$$\hat{\mathbf{n}}_2^T A (\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2) = \sin\varphi \cos\psi + \cos\varphi \sin\psi \cos(\vartheta - \lambda), \tag{43}$$

$$(\hat{\mathbf{n}}_2 \times \hat{\mathbf{n}}_3)^T A \hat{\mathbf{n}}_2 = \cos\varphi \sin\psi + \sin\varphi \cos\psi \cos(\vartheta - \lambda), \tag{44}$$

and

$$(\hat{\mathbf{n}}_2 \times \hat{\mathbf{n}}_3)^T A (\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2) = \sin\varphi \sin\psi - \cos\varphi \cos\psi \cos(\vartheta - \lambda). \tag{45}$$

Now we can either find φ from equation (34) and ψ from

$$\psi = \mathrm{ATAN2}\Big[\cos\varphi(\hat{\mathbf{n}}_2 \times \hat{\mathbf{n}}_3)^T A\hat{\mathbf{n}}_2 + \sin\varphi(\hat{\mathbf{n}}_2 \times \hat{\mathbf{n}}_3)^T A(\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2), \quad \cos\varphi\, \hat{\mathbf{n}}_2^T A\hat{\mathbf{n}}_2 + \sin\varphi\, \hat{\mathbf{n}}_2^T A(\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2)\Big], \tag{46}$$

or, alternatively, we can find ψ from equation (35) and then φ from

$$\varphi = \mathrm{ATAN2}\Big[\cos\psi\, \hat{\mathbf{n}}_2^T A(\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2) + \sin\psi(\hat{\mathbf{n}}_2 \times \hat{\mathbf{n}}_3)^T A(\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2), \quad \cos\psi\, \hat{\mathbf{n}}_2^T A\hat{\mathbf{n}}_2 + \sin\psi\, (\hat{\mathbf{n}}_2 \times \hat{\mathbf{n}}_3)^T A\hat{\mathbf{n}}_2\Big]. \tag{47}$$

Note that both equations (46) and (47) are well behaved for all values of ϑ. The use of one of these alternatives guarantees that the well-defined linear combination of φ and ψ is determined accurately even when the solution to equation (34) or (35) loses numerical significance. However, these methods are more computationally expensive than using equations (34) and (35) together, and it is best in practice to choose a set of Euler axes for which the gimbal-lock phenomenon will not be encountered.

Discussion

We have shown that the Euler angles can be generalized to encompass sequences of rotations about any three axes subject to the constraint that axes of successive rotations be perpendicular. Thus the second rotation axis must be orthogonal to both the first and the third, but the angle between the first and third axes is arbitrary. This angle, the 'new angle' promised in the title, can take on any value rather than being restricted to the values 0 or $\pm\,\pi/2$ as in the conventional Euler angle sequences. Kinematic equations have been derived for the generalized Euler angles, as well as equations for extracting these angles from the rotation matrix. The generalized Euler angles have the same 'gimbal lock' singularity as the conventional angle sets. Means for circumventing this problem developed for the conventional cases have been extended to the generalized Euler angles.

All the equations in this paper can be applied to the conventional Euler angle sets in a straightforward fashion, so a side benefit of this work has been to supply universal formulas applicable to all Euler angle parameterizations, conventional and generalized.

Acknowledgement

We are grateful to R. V. F. Lopes for a careful reading of an earlier version of this work.

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