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CONSISTENT ESTIMATION OF SPACECRAFT SENSOR ALIGNMENTS

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Abstract

Simple and statistically correct algorithms are developed for batch estimation of spacecraft sensor alignments from prelaunch and inflight data without the need to compute the spacecraft attitude or angular velocity. These algorithms permit the estimation of sensor alignments in a framework free of unknown dynamical variables. In actual mission implementation, algorithms such as those presented here are usually better behaved and more efficient than those which must compute sensor alignments simultaneously with the spacecraft attitude, say, by means of a Kalman filter. In particular, these algorithms are less sensitive to data dropouts of long duration, and the derived measurements used in the attitude-independent algorithm usually make data checking and editing of outliers much simpler than would be the case in the filter. An estimator for the launch-shock error levels is also developed and the effect of unobservable launch shock on the misalignment estimates is studied. The algorithms are applied to a realistic simulated example which approximates actual missions.

INTRODUCTION

Sensor Referenced Measurements

A spacecraft line-of-sight sensor such as a vector Sun sensor or star tracker measures a direction, $\hat{\mathbf{U}}_{i,k}$, in sensor coordinates, defined to be directed outward from the sensor, which is describable statistically as

$$\hat{\mathbf{U}}_{i,k} = \hat{\mathbf{U}}_{i,k}^{\text{true}} + \Delta\hat{\mathbf{U}}_{i,k} \quad , \tag{1}$$

where $\hat{\mathbf{U}}_{i,k}^{\text{true}}$ is the true value of the direction and $\Delta\hat{\mathbf{U}}_{i,k}$ is the measurement noise. Here i is the sensor index, $i = 1, \ldots, n$, and k is the temporal index, $k = 1, \ldots, N$. We assume that $\Delta\hat{\mathbf{U}}_{i,k}$ is Gaussian, zero-mean, and white, with covariance $R_{\hat{U}_{i,k}}$. In more compact notation

$$\Delta\hat{\mathbf{U}}_{i,k} \sim \mathcal{N}(\mathbf{0}, R_{\hat{U}_{i,k}}) \quad . \tag{2}$$

We assume more generally, in fact, that the measurements from different sensors are statistically independent. Because the observations are constrained to be unit vectors, $R_{\hat{U}_{i,k}}$ must be singular. In particular,

$$R_{\hat{U}_{i,k}} \hat{\mathbf{U}}_{i,k}^{\text{true}} = \mathbf{0} \quad . \tag{3}$$

Equations (2) and (3) can be true only to lowest order in R. Since R is generally quite small, this level of approximation will be adequate for the purpose of alignment estimation.

Body-Referenced Vectors and Alignments

If $\hat{\mathbf{W}}_{i,k}$ denotes the measured direction in the spacecraft body frame, then the alignment matrix, S_i, is the proper orthogonal matrix defined by

$$\hat{\mathbf{W}}_{i,k} = S_i \hat{\mathbf{U}}_{i,k} \quad , \tag{4}$$

and, therefore,

$$\hat{\mathbf{W}}_{i,k} = S_i \hat{\mathbf{U}}_{i,k}^{\text{true}} + S_i \Delta\hat{\mathbf{U}}_{i,k} \quad , \tag{5}$$

$$\equiv \hat{\mathbf{W}}_{i,k}^{\text{true}} + \Delta\hat{\mathbf{W}}_{i,k} \quad . \tag{6}$$

In general, the alignment matrix S_i is not known exactly. Instead, what is known is S_i^o, the alignment matrix determined by the prelaunch alignment calibration. Thus, we are led to define the misalignment matrix, M_i, according to

$$S_i = M_i S_i^o \quad . \tag{7}$$

M_i is necessarily orthogonal. We define the misalignment vectors, $\boldsymbol{\theta}_i$, according to

$$M_i \equiv e^{[[\boldsymbol{\theta}_i]]} \quad ,$$
$$= I + [[\boldsymbol{\theta}_i]] + O(|\boldsymbol{\theta}_i|^2) \quad , \tag{8}$$

where $e^{\{\cdot\}}$ denotes here matrix exponentiation, and $[[\boldsymbol{\theta}]]$ denotes the usual antisymmetric matrix,

$$[[\boldsymbol{\theta}]] \equiv \begin{bmatrix} 0 & \theta_3 & -\theta_2 \\ -\theta_3 & 0 & \theta_1 \\ \theta_2 & -\theta_1 & 0 \end{bmatrix} \quad . \tag{9}$$

As a rule, we will keep only first-order terms. The measurement equation may now be written

$$\hat{\mathbf{U}}_{i,k} = S_i^{oT} M_i^T \hat{\mathbf{W}}_{i,k}^{\text{true}} + \Delta\hat{\mathbf{U}}_{i,k} \quad . \tag{10}$$

PRELAUNCH ALIGNMENT CALIBRATION

Prior to launch, the alignments of the attitude sensors are measured on the ground. Generally, optical cubes, whose adjacent faces are very nearly perpendicular, are affixed to each sensor and the sensor is calibrated with respect to its optical cube. One of these cubes affixed to the spacecraft payload or structure, called the primary reference cube, defines the spacecraft body axes and provides the reference for all other cube alignments. At NASA Goddard Space Flight Center, for example [1, 2], alignment calibration is accomplished by sighting the normals to two faces of each cube. If these two normals are denoted by $\hat{\mathbf{n}}_{i,m}$, $m = 1, 2$, then we may define a triad of orthonormal unit vectors according to

$$\hat{\mathbf{e}}_{i,1} \equiv \hat{\mathbf{n}}_{i,1} \quad , \tag{11a}$$

$$\hat{\mathbf{e}}_{i,2} \equiv \hat{\mathbf{n}}_{i,1} \times \hat{\mathbf{n}}_{i,2} / |\hat{\mathbf{n}}_{i,1} \times \hat{\mathbf{n}}_{i,2}| \quad , \tag{11b}$$

$$\hat{\mathbf{e}}_{i,3} \equiv \hat{\mathbf{e}}_{i,1} \times \hat{\mathbf{e}}_{i,2} \quad . \tag{11c}$$

The prelaunch alignment matrix is then given by

$$(S_i^o)_{\ell\ell'} = \hat{\mathbf{e}}_{o,\ell}^T \, \hat{\mathbf{e}}_{i,\ell'} \tag{12}$$

This is the way in which the prelaunch alignments have been determined for the Solar Maximum Mission (SMM), Magsat, the International Ultraviolet Explorer (IUE), and numerous other spacecraft. Equation (12) is simply the TRIAD algorithm [3] for attitude determination applied to alignment estimation.

To compute a covariance matrix for the prelaunch alignment estimates we assume that the measured normals may be described statistically by

$$\hat{\mathbf{n}}_{i,\ell} = \hat{\mathbf{n}}_{i,\ell}^{\text{true}} + \Delta\hat{\mathbf{n}}_{i,\ell} \quad . \tag{13}$$

The $\Delta\hat{\mathbf{n}}_{i,\ell}$ are assumed to be Gaussian, zero-mean and with covariance given by

$$E\{\Delta\hat{\mathbf{n}}_{i,\ell} \Delta\hat{\mathbf{n}}_{i',\ell'}^T\} = \delta_{ii'} \, \delta_{\ell\ell'} \, \sigma_p^2 \left(I_{3\times 3} - \hat{\mathbf{n}}_{i,\ell}^{\text{true}} \, \hat{\mathbf{n}}_{i,\ell}^{\text{true}\,T}\right) \quad . \tag{14}$$

The interpretation of equation (14) is that each unit normal has errors which are distributed with axial symmetry about the true value of the normal with angular standard deviation σ_p. As a consequence of the errors in the measured normals we write

$$\boldsymbol{\theta}_i^*(\text{prelaunch}) = 0 = \boldsymbol{\theta}_i^{\text{prelaunch}} + \Delta\boldsymbol{\theta}_i^*(\text{prelaunch}) \quad , \tag{15}$$

where the $\Delta\boldsymbol{\theta}_i^*(\text{prelaunch})$, $i = 1, \ldots, n$, are the errors in the prelaunch estimates of the misalignment vectors, which are then also zero-mean and Gaussian. In general, a caret will be used to denote unit vectors; estimates (and estimators) will be denoted by an asterisk. We write

$$E\{\Delta\boldsymbol{\theta}_i^*(\text{prelaunch})\} = 0 \quad , \tag{16a}$$

$$E\{\Delta\boldsymbol{\theta}_i^*(\text{prelaunch}) \, \Delta\boldsymbol{\theta}_j^{*\,T}(\text{prelaunch})\} = P_{ij}(\text{prelaunch}) \quad , \tag{16b}$$

The prelaunch alignment covariance matrix based on these statistical assumptions is [4]

$$P_{ij}(\text{prelaunch}) = \sigma_p^2 \left(1 + \delta_{ij}\right) I_{3\times 3} \quad , \tag{17}$$

showing that the prelaunch alignments are correlated. The simplicity of equation (17) is due in part to the fact that because of equation (7) all misalignments are referred, within small errors, to the same reference frame. Thus, the alignment matrices themselves do not enter equation (17). From the observed repeatability of the measurements we are led to assign to σ_p a value (in radian equivalent) consistent with

$$\sigma_p \approx 3.5. \text{ arc sec} \quad . \tag{18}$$

LAUNCH SHOCK

The values of the spacecraft sensor alignments change after the launch of the spacecraft from their prelaunch values. This change is due to a variety of phenomena including vibration, changes in the distribution of temperatures in the spacecraft (which are responsible both for deformation of the spacecraft and the sensors as well as for changes in the sensor electronics), and zero-gravity effects. We lump all of these effects under the general heading of launch shock. Since there is little similarity among spacecraft, it is difficult to obtain a truly representative general statistical characterization of launch shock. For the sake of simplicity we write

$$\Theta^{\text{inflight}} = \Theta^{\text{prelaunch}} + \Delta\Theta^{\text{launch-shock}} \quad , \tag{19}$$

where

$$\Theta \equiv [\boldsymbol{\theta}_1^T, \ldots, \boldsymbol{\theta}_n^T]^T \quad , \tag{20}$$

and assume that

$$\Delta\Theta^{\text{launch-shock}} \sim \mathcal{N}(0, \, Q_{\Theta\Theta}^{\text{launch-shock}}) \quad . \tag{21}$$

In order to make the distinction between (possibly random) physical variables and their estimates more visible, we have written the labels for the former as verbal superscripts and the labels of the latter in parentheses. Based on this model, the *a priori* maximum likelihood estimate of the inflight sensor misalignments and their covariance is given by

$$\Theta^*(-) = \Theta^*(\text{prelaunch}) = 0 \quad , \tag{22}$$

$$P_{\Theta\Theta}(-) = P_{\Theta\Theta}(\text{prelaunch}) + Q_{\Theta\Theta}^{\text{launch-shock}} \quad . \tag{23}$$

Since the launch-shock introduces a change in the physical alignments we must now distinguish between $\Theta^{\text{prelaunch}}$ and Θ^{inflight}. To simplify the notation, when Θ appears without a verbal superscript, it generally denotes Θ^{inflight}.

INFLIGHT ALIGNMENT CALIBRATION

Dependence of the Measurements on the Attitude

If $\hat{\mathbf{V}}_{i,k}$ denotes the reference vector, i.e., the representation of the measured vector in the primary reference system, then the attitude matrix A_k is defined according to

$$\hat{\mathbf{W}}_{i,k}^{\text{true}} = A_k \, \hat{\mathbf{V}}_{i,k}^{\text{true}} \quad , \tag{24}$$

whence,

$$\hat{\mathbf{W}}_{i,k} = A_k \hat{\mathbf{V}}_{i,k} + \Delta\hat{\mathbf{W}}_{i,k} - A_k \Delta\hat{\mathbf{V}}_{i,k} \quad , \tag{25}$$

where $\Delta\hat{\mathbf{V}}_{i,k}$ is the uncertainty in the reference vector, which we assume to be Gaussian, zero-mean, and white. Hence,

$$E\{\Delta\hat{\mathbf{V}}_{i,k} \, \Delta\hat{\mathbf{V}}_{i'k'}^T\} = \delta_{ii'} \, \delta_{kk'} \, R_{\hat{\mathbf{V}}_{i,k}} \quad . \tag{26}$$

From this it follows that the actual sensor measurements are related to the reference vectors by

$$\hat{\mathbf{U}}_{i,k} = S_i^T A_k \hat{\mathbf{V}}_{i,k} + \Delta\hat{\mathbf{U}}_{i,k} - S_i^T A_k \Delta\hat{\mathbf{V}}_{i,k} \quad . \tag{27}$$

We note immediately from equation (27) that the values of the measurement vectors are unchanged by the simultaneous transformations

$$S_i \to T\,S_i \quad , \quad i = 1, \ldots, n \quad , \tag{28a}$$
$$A_k \to T\,A_k \quad , \quad k = 1, \ldots, N \quad , \tag{28b}$$

where T is an arbitrary proper orthogonal matrix. Thus, it is impossible from sensor measurements to distinguish a common misalignment of the sensors from a change in the attitude. It is, therefore, impossible to estimate the sensor alignments and the attitude unambiguously from the spacecraft sensor measurements alone.

In terms of the misalignments, equation (27) becomes

$$\hat{\mathbf{U}}_{i,k} = S_i^{oT} M_i^T A_k \hat{\mathbf{V}}_{i,k} + \Delta\hat{\mathbf{U}}_{i,k} - S_i^{oT} M_i^T A_k \Delta\hat{\mathbf{V}}_{i,k} \quad . \tag{29}$$

The Attitude-Independent Measurements

Equation (29) is the starting point for processing the inflight data. We begin by defining an "uncalibrated" body-referenced observation vector, $\hat{\mathbf{W}}^o_{i,k}$, according to

$$\hat{\mathbf{W}}^o_{i,k} \equiv S_i^o\, \hat{\mathbf{U}}_{i,k} = M_i^T\, \mathbf{W}_{i,k} \quad , \tag{30}$$

so that

$$\hat{\mathbf{W}}^o_{i,k} = M_i^T A_k \hat{\mathbf{V}}_{i,k} + \Delta\hat{\mathbf{W}}^o_{i,k} \quad , \tag{31}$$

with

$$\Delta\hat{\mathbf{W}}^o_{i,k} \sim \mathcal{N}(0, R_{\mathbf{W}^o_{i,k}}) \quad , \tag{32}$$

and

$$R_{\mathbf{W}^o_{i,k}} = S_i^o\, R_{\hat{\mathbf{U}}_{i,k}}\, S_i^{oT} + M_i^T A_k R_{\hat{\mathbf{V}}_{i,k}} A_k^T M_i \quad , \tag{33}$$

If for $i \neq j$ we define effective scalar measurements,

$$z_{ij,k} \equiv \hat{\mathbf{W}}^o_{i,k} \cdot \hat{\mathbf{W}}^o_{j,k} - \hat{\mathbf{V}}_{i,k} \cdot \hat{\mathbf{V}}_{j,k} \quad , \tag{34}$$

then, to first order in θ_i, θ_j, and the measurement noise terms we have that

$$z_{ij,k} = (\hat{\mathbf{W}}^o_{i,k} \times \hat{\mathbf{W}}^o_{j,k}) \cdot (\theta_i - \theta_j) + \Delta z_{ij,k} \quad , \tag{35}$$

with

$$\Delta z_{ij,k} \approx \hat{\mathbf{W}}^o_{i,k} \cdot \Delta\hat{\mathbf{W}}^o_{j,k} + \hat{\mathbf{W}}^o_{j,k} \cdot \Delta\hat{\mathbf{W}}^o_{i,k} \quad . \tag{36}$$

which are independent of the attitude. The derived measurements, which are the observed cosine errors, are independent of the attitude to first order in the misalignments. To lowest order in the misalignments and the measurement noise, the derived measurement errors satisfy

$$E\{\Delta z_{ij,k}\} = 0 \quad , \tag{37a}$$
$$E\{\Delta z_{ij,k}^2\} = E_k(i \mid j \mid i) + E_k(j \mid i \mid j) \quad , \tag{37b}$$
$$E\{\Delta z_{ij,k} \Delta z_{i\ell,k}\} = E_k(j \mid i \mid \ell) \quad , \tag{37c}$$
$$E\{\Delta z_{ij,k} \Delta z_{\ell m,k}\} = 0 \quad , \tag{37d}$$

where i, j, ℓ, and m above denote distinct indices and $E_k(i \mid j \mid \ell)$ is given by

$$E_k(i \mid j \mid \ell) \equiv \hat{\mathbf{W}}^{oT}_{i,k}\, R_{\mathbf{W}^o_{j,k}}\, \hat{\mathbf{W}}^o_{\ell,k} \quad . \tag{38}$$

Note that $z_{ij,k}$ is symmetric in the indices i and j.

The Redundancy Problem

The measurements $z_{ij,k}$, $i < j$, cannot all be independent. If there are n sensors, each measuring a unit vector, then there are only $2n$ equivalent independent scalar measurements, while there are $n(n-1)/2$ possible $z_{ij,k}$ with $i < j$. Since three combinations of the $\hat{\mathbf{W}}^o_{i,k}$ are necessary to determine the attitude, and the $z_{ij,k}$ are by explicit construction attitude-independent, there can only be $2n-3$ statistically independent $z_{ij,k}$. Clearly for $n > 3$

$$2n - 3 < \frac{n(n-1)}{2} \quad . \tag{39}$$

A set of $2n-3$ independent measurements can be obtained as

$$\{z_{1j,k}, j = 2, \ldots, n\,;\; z_{2j,k}, j = 3, \ldots, n\}$$

provided that $\hat{\mathbf{W}}^o_{1,k}$ and $\hat{\mathbf{W}}^o_{2,k}$ are not collinear nor are they coplanar or nearly coplanar to any of the remaining measurements [4]. When the measurement vectors are all coplanar, the derived scalar-product measurements are not sensitive to misalignment components out of the plane and must be supplement by scalar-triple-product measurements [4] of the form

$$z_{ij\ell,k} \equiv \hat{\mathbf{W}}^o_{i,k} \cdot (\hat{\mathbf{W}}^o_{j,k} \times \hat{\mathbf{W}}^o_{\ell,k}) \quad , \tag{40}$$

leading to a far greater redundancy.

The Inflight Estimator

If we define

$$\mathbf{Z}_k \equiv [z_{12,k}, \ldots, z_{1n,k}, z_{23,k}, \ldots, z_{2n,k}]^T \quad , \tag{41}$$

we may write

$$\mathbf{Z}_k = H_k \Theta + \Delta\mathbf{Z}_k \quad , \tag{42}$$

Thus, $\Delta\mathbf{Z}_k$ is a white Gaussian sequence with covariance matrix $P_{\mathbf{Z}_k}$. The matrices H_k and $P_{\mathbf{Z}_k}$ are obtained directly from equations (35) through (38). The *a posteriori* inflight estimate of the misalignments, $\Theta^*(+)$, together with the *a posteriori* estimate error covariance, $P_{\Theta\Theta}(+)$, may be obtained straightforwardly by maximum likelihood estimation [5, 6]. The negative-log-likelihood function (the negative of the logarithm of the joint probability density function of the measurements and the parameters), which in maximum likelihood estimation serves as a cost function, is simply

$$\begin{aligned} J_\Theta(\Theta) = \frac{1}{2}\left[\Theta^T P_{\Theta\Theta}^{-1}(-)\,\Theta + \log\det P_{\Theta\Theta}(-) + 3n \log 2\pi\right] \\ + \tfrac{1}{2}\sum_{k=1}^{N}\left[(\mathbf{Z}_k - H_k\Theta)^T P_{\mathbf{Z}_k}^{-1}(\mathbf{Z}_k - H_k\Theta)\right. \\ \left. + \log\det P_{\mathbf{Z}_k} + (2n_k - 3)\log 2\pi\right] \end{aligned} \tag{43}$$

where $2n_k - 3$ is the dimension of $\mathbf{Z}_k$ if there are fewer than the full complement of sensors active at a given time t_k. Minimizing $J_\Theta(\Theta)$ leads to the usual normal equations:

$$P_{\Theta\Theta}^{-1}(+)\,\Theta^*(+) = \sum_{k=1}^{N} H_k^T P_{\mathbf{Z}_k}^{-1} \mathbf{Z}_k \quad , \tag{44}$$

$$P_{\Theta\Theta}^{-1}(+) = P_{\Theta\Theta}^{-1}(-) + \sum_{k=1}^{N} H_k^T P_{\mathbf{Z}_k}^{-1} H_k \quad . \tag{45}$$

Since the measurements are assumed to be Gaussian, we have already written the *a posteriori* Fisher information matrix as

$P_{\Theta\Theta}^{-1}(+)$, since in this case it is equal to the inverse of the estimate error covariance matrix even for small samples. When there are only a few sensors, say three or four, which are typically far from having parallel boresights among them, the above formalism (augmented by a suitable model for $P_{\Theta\Theta}(-)$) should be adequate for alignment estimation and is certainly simple to implement. Complications which arise from the combinatorics entailed in having a large number of sensors or from nearly parallel measurements are treated elsewhere [4, 7].

An important complication that arises in evaluating Eqs. (44) and (45) is that $P_{\Theta\Theta}(-)$ depends on the launch-shock process whose parameters are generally unknown and must be inferred also from the inflight data. $\Theta^*(+)$ is a very inconvenient quantity to use to estimate the launch-shock parameters since it depends on them directly.

ESTIMATION OF LAUNCH-SHOCK ERROR LEVELS

We develop now a methodology for estimating the launch-shock covariance parameters from inflight data. Since so little data is available to characterize the launch shock we might choose to make the simplest model possible for the launch-shock error covariance matrix, namely,

$$Q_{\Theta\Theta}^{\text{launch-shock}} = q\, I_{3n\times 3n} \quad . \tag{46}$$

In some cases, say when the primary reference cube and some of the sensors are mounted to an extremely rigid instrument plate (optical bench) while other sensors are scattered about the spacecraft, we might wish to propose a smaller value, q_1, for the sensors on the instrument plate and a larger value q_2, for the other sensors, or allow the launch shock effects of a number of sensors jointly mounted on some distant but rigid surface to be highly correlated. However, it should be considered first that not all of the apparent misalignment is due to geometric distortion of the spacecraft. Secondly, it must be kept in mind that the number of sensors in the spacecraft is limited, so that there are only $3n - 3$ quantities which may be used to estimate the parameters of the launch shock. Hence, while it may be reasonable to simulate detailed launch-shock effects before launch, it is a hopeless task to try to estimate the parameters of a very detailed model from inflight data. For the sake of generality, however, we write

$$Q_{\Theta\Theta}^{\text{launch-shock}} = Q_{\Theta\Theta}^{\text{launch-shock}}(\mathbf{q}) \quad , \tag{47}$$

where $\mathbf{q}$ is the vector of launch-shock parameters.

We could estimate the launch-shock covariance parameters simply by looking at the distribution of the inflight estimates of the misalignments $\boldsymbol{\theta}_i^*(+)$, $i = 1, \ldots, n$, but this would be difficult owing to the complicated dependence of $\Theta^*(+)$ and $P_{\Theta\Theta}(+)$ on $\mathbf{q}$. For this reason we examine instead the set of relative misalignments defined as

$$\boldsymbol{\psi}_i \equiv \boldsymbol{\theta}_i - \boldsymbol{\theta}_1 \quad , \quad i = 2, \ldots, n \quad . \tag{48}$$

This reduced set of parameters is observable from inflight data alone. The total relative misalignment vector is defined as

$$\Psi \equiv [\, \boldsymbol{\psi}_2^T, \boldsymbol{\psi}_3^T, \ldots, \boldsymbol{\psi}_n^T \,]^T \quad , \tag{49a}$$

$$= F\,\Theta \quad , \tag{49b}$$

where

$$F = \begin{bmatrix} -I_{3\times3} & I_{3\times3} & 0_{3\times3} & \cdots & 0_{3\times3} \\ -I_{3\times3} & 0_{3\times3} & I_{3\times3} & \cdots & 0_{3\times3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -I_{3\times3} & 0_{3\times3} & 0_{3\times3} & \cdots & I_{3\times3} \end{bmatrix} \quad . \tag{50}$$

In terms of the relative misalignments the measurement vector may be written as

$$\mathbf{Z}_k = H_k'\,\Psi + \Delta\mathbf{Z}_k \quad , \tag{51}$$

where the measurement sensitivity matrix, H_k, has been partitioned as

$$H_k = [\, h_{1k} \mid H_k' \,] \quad . \tag{52}$$

We may estimate Ψ based on the inflight data alone, not taking into account the *a priori* model. The negative-log-likelihood function for this prior-free estimate of Ψ is simply

$$J_\Psi^{\text{prior-free}}(\Psi) = \tfrac{1}{2} \sum_{k=1}^{N} \Big[(\mathbf{Z}_k - H_k'\Psi)^T P_{\mathbf{Z}_k}^{-1} (\mathbf{Z}_k - H_k'\Psi) + \log\det P_{\mathbf{Z}_k} + (2n_k - 3)\log 2\pi \Big] \tag{53}$$

Minimizing $J_\Psi^{\text{prior-free}}(\Psi)$ over Ψ leads to the equations

$$P_{\Psi\Psi}^{-1}(\text{prior-free})\,\Psi^*(\text{prior-free}) = \sum_{k=1}^{N} H_k'^T P_{\mathbf{Z}_k}^{-1} \mathbf{Z}_k \quad , \tag{54}$$

with

$$P_{\Psi\Psi}^{-1}(\text{prior-free}) = \sum_{k=1}^{N} H_k'^T P_{\mathbf{Z}_k}^{-1} H_k' \quad . \tag{55}$$

The *a posteriori* estimate of the *absolute* misalignment vector can be easily recovered from the prior-free estimate of the *relative* misalignment vector according to

$$P_{\Theta\Theta}^{-1}(+)\,\Theta^*(+) = F^T P_{\Psi\Psi}^{-1}(\text{prior-free})\,\Psi^*(\text{prior-free}) \quad , \tag{56}$$

and

$$P_{\Theta\Theta}^{-1}(+) = P_{\Theta\Theta}^{-1}(-) + F^T P_{\Psi\Psi}^{-1}(\text{prior-free})\,F \quad . \tag{57}$$

Note, however, that equations (56) and (57) can only be implemented after $\mathbf{q}$ has been determined, since $P_{\Theta\Theta}(-)$ depends on $\mathbf{q}$.

To determine $\mathbf{q}^*$ we note that

$$\Psi^*(\text{prior-free}) = \Psi^{\text{inflight}} + \Delta\Psi^*(\text{prior-free}) \quad , \tag{58}$$

and

$$\Psi^{\text{inflight}} = \Psi^{\text{prelaunch}} + \Delta\Psi^{\text{launch-shock}} \quad ,$$
$$= F\,\Theta^{\text{prelaunch}} + F\Delta\Theta^{\text{launch-shock}} \quad . \tag{59}$$

From Eq. (15)

$$\Theta^{\text{prelaunch}} = \Theta^*(\text{prelaunch}) - \Delta\Theta^*(\text{prelaunch}) \quad ,$$
$$= -\Delta\Theta^*(\text{prelaunch}) \quad . \tag{60}$$

and, therefore,

$$E\{\,\Psi^*(\text{prior-free})\,\} = \mathbf{0} \quad , \tag{61a}$$

$$E\{\,\Psi^*(\text{prior-free})\,\Psi^{*T}(\text{prior-free})\,\} = P_{\Psi\Psi}(\text{total}) \quad , \tag{61b}$$

where

$$P_{\Psi\Psi}(\text{total}) \equiv P_{\Psi\Psi}(\text{prelaunch}) + Q_{\Psi\Psi}^{\text{ls}}(\mathbf{q}) + P_{\Psi\Psi}(\text{prior-free}) \quad , \tag{61c}$$

and

$$P_{\Psi\Psi}(\text{prelaunch}) \equiv F\, P_{\Theta\Theta}(\text{prelaunch}) F^T \quad , \tag{62a}$$

$$P_{\Psi\Psi}(\text{prior-free}) \equiv F\, P_{\Theta\Theta}(\text{prior-free}) F^T \quad , \tag{62b}$$

$$Q^{\text{ls}}_{\Psi\Psi}(\mathbf{q}) \equiv F\, Q^{\text{launch-shock}}_{\Theta\Theta}\, F^T \quad . \tag{62c}$$

Thus, the negative-log-likelihood function for q given Ψ^*(prior-free) is

$$J^{\text{prior-free}}_q(\mathbf{q}) = \tfrac{1}{2}\left[\Psi^{*T}(\text{prior-free}) P^{-1}_{\Psi\Psi}(\text{total})\, \Psi^*(\text{prior-free}) + \log\det P_{\Psi\Psi}(\text{total}) + (3n-3)\log 2\pi\right] \quad , \tag{63}$$

which depends on q only through $Q^{\text{ls}}_{\Psi\Psi}(\mathbf{q})$. We could, as indicated earlier, also have constructed a negative-log-likelihood function based on $\Psi^*(+)$ or even $\Theta^*(+)$, but the dependence on q would have been more complicated because q enters explicitly in the calculation of these quantities. Note that equations (54) and (55) do not depend explicitly on q. Thus, $\mathbf{q}^*$ is a solution of

$$\frac{\partial J^{\text{prior-free}}_q}{\partial \mathbf{q}}(\mathbf{q}^*) = \tfrac{1}{2}\left\{\left[-\Psi^{*T}(\text{prior-free})\, P^{-1}_{\Psi\Psi}(\text{total}) \times \frac{\partial Q^{\text{ls}}_{\Psi\Psi}(\mathbf{q})}{\partial \mathbf{q}}\, P^{-1}_{\Psi\Psi}(\text{total})\, \Psi^*(\text{prior-free})\right] + \text{tr}\left(P_{\Psi\Psi}(\text{total})\frac{\partial Q^{\text{ls}}_{\Psi\Psi}(\mathbf{q})}{\partial \mathbf{q}}\right)\right\} = 0 \quad , \tag{64}$$

which must be solved iteratively for $\mathbf{q}^*$. Asymptotically (i.e., as $n \to \infty$) the variance of the estimate error for $\mathbf{q}^*$ is given by

$$P^{-1}_{qq} = \tfrac{1}{2}\,\text{tr}\left[\left(P^{-1}_{\Psi\Psi}(\text{total})\,\frac{\partial Q^{\text{ls}}_{\Psi\Psi}(\mathbf{q})}{\partial \mathbf{q}}\right)^2\right] \quad . \tag{65}$$

As an example, consider the simplest parameterization of $Q^{\text{launch-shock}}_{\Theta\Theta}$ given by Eq. (46) and the parameterization of the prelaunch alignment estimate error covariance given by Eq. (17). If we assume that sufficient data has been collected that

$$P_{\Psi\Psi}(\text{prelaunch}) << Q^{\text{ls}}_{\Psi\Psi}(\mathbf{q}) \quad , \tag{66a}$$

$$P_{\Psi\Psi}(\text{prior-free}) << Q^{\text{ls}}_{\Psi\Psi}(\mathbf{q}) \quad , \tag{66b}$$

then Eqs. (64) and (65) reduce to

$$q^* \approx \frac{1}{3(n-1)}\left(\sum_{i=2}^{n} |\psi^*_i(\text{prior-free})|^2 - \frac{1}{n}\left|\sum_{i=2}^{n}\psi^*_i(\text{prior-free})\right|^2\right) \quad , \tag{67}$$

and

$$P_{qq} \approx \frac{2q^2}{3(n-1)} \quad . \tag{68}$$

A NUMERICAL EXAMPLE

We illustrate the power of these methods with a numerical example. Consider a typical spacecraft equipped with three vector sensors each with an accuracy of 10. arc sec/axis and an effective (weighted) field of view of ±10. deg/axis. We assume the sensor errors to be well represented by the QUEST measurement model [3], which has been used in the attitude determination algorithm for several spacecraft. We may write this measurement model as

$$E\{\Delta\hat{\mathbf{U}}_{i,k}\,\Delta\hat{\mathbf{U}}^T_{i,k}\} = \sigma^2_{\hat{\mathbf{U}}_{i,k}}\left(I - \hat{\mathbf{U}}^{\text{true}}_{i,k}\,\hat{\mathbf{U}}^{\text{true}\,T}_{i,k}\right) \quad , \tag{69}$$

for the sensor measurements and

$$E\{\Delta\hat{\mathbf{V}}_{i,k}\,\Delta\hat{\mathbf{V}}^T_{i,k}\} = \sigma^2_{\hat{\mathbf{V}}_{i,k}}\left(I - \hat{\mathbf{V}}^{\text{true}}_{i,k}\,\hat{\mathbf{V}}^{\text{true}\,T}_{i,k}\right) \quad , \tag{70}$$

for the reference vectors. Thus, to lowest order in the misalignments

$$E\{\Delta\hat{\mathbf{W}}^o_{i,k}\,\Delta\hat{\mathbf{W}}^{oT}_{i,k}\} = \sigma^2_{i,k}\left(I - \hat{\mathbf{W}}^o_{i,k}\,\hat{\mathbf{W}}^{oT}_{i,k}\right) \quad , \tag{71}$$

with

$$\sigma^2_{i,k} = \sigma^2_{\hat{\mathbf{U}}_{i,k}} + \sigma^2_{\hat{\mathbf{V}}_{i,k}} \quad . \tag{72}$$

Substituting these expressions into equation (38) it follows for the QUEST model that

$$E_k(i\mid j\mid \ell) = \sigma^2_{j,k}\,(\hat{\mathbf{W}}^o_{i,k}\times\hat{\mathbf{W}}^o_{j,k})\cdot(\hat{\mathbf{W}}^o_{\ell,k}\times\hat{\mathbf{W}}^o_{j,k}) \quad , \tag{73}$$

Note that the approximation of the QUEST measurement model lies in the form of the covariance matrix of the measurement noise, not in the nonrandom part of the measurement.

Thus, the measurement is described by

$$\mathbf{Z}_k = \begin{bmatrix} z_{23,k} \\ z_{31,k} \\ z_{12,k}\end{bmatrix} = H_k\,\Theta + \Delta\mathbf{Z}_k \quad , \tag{74}$$

where the reordering of the components and sign changes will serve to give the measurement vector a cyclic symmetry and simplify later calculations. The sensitivity matrix, H_k, is given now by

$$H_k = \begin{bmatrix} 0^T & (\mathbf{W}^o_{2,k}\times\mathbf{W}^o_{3,k})^T & -(\mathbf{W}^o_{2,k}\times\mathbf{W}^o_{3,k})^T \\ -(\mathbf{W}^o_{3,k}\times\mathbf{W}^o_{1,k})^T & 0^T & (\mathbf{W}^o_{3,k}\times\mathbf{W}^o_{1,k})^T \\ (\mathbf{W}^o_{1,k}\times\mathbf{W}^o_{2,k})^T & -(\mathbf{W}^o_{1,k}\times\mathbf{W}^o_{2,k})^T & 0^T \end{bmatrix} \tag{75}$$

and, therefore,

$$P_{\mathbf{Z},k} = \begin{bmatrix} d_{23,k} & f_{23,31,k} & f_{23,12,k} \\ f_{23,31,k} & d_{31,k} & f_{31,12,k} \\ f_{23,12,k} & f_{31,12,k} & d_{12,k}\end{bmatrix} \quad , \tag{76}$$

where

$$d_{ij,k} = (\sigma^2_{i,k} + \sigma^2_{j,k})|\hat{\mathbf{W}}^o_{i,k}\times\hat{\mathbf{W}}^o_{j,k}|^2 \quad , \tag{77a}$$

$$f_{23,31,k} = -\sigma^2_{3,k}\,(\hat{\mathbf{W}}^o_{2,k}\times\hat{\mathbf{W}}^o_{3,k})\cdot(\hat{\mathbf{W}}^o_{3,k}\times\hat{\mathbf{W}}^o_{1,k}) \quad , \tag{77b}$$

$$f_{23,12,k} = -\sigma^2_{2,k}\,(\hat{\mathbf{W}}^o_{2,k}\times\hat{\mathbf{W}}^o_{3,k})\cdot(\hat{\mathbf{W}}^o_{1,k}\times\hat{\mathbf{W}}^o_{2,k}) \quad , \tag{77c}$$

$$f_{31,12,k} = -\sigma^2_{1,k}\,(\hat{\mathbf{W}}^o_{3,k}\times\hat{\mathbf{W}}^o_{1,k})\cdot(\hat{\mathbf{W}}^o_{1,k}\times\hat{\mathbf{W}}^o_{2,k}) \quad . \tag{77d}$$

We assume also that the prelaunch alignment calibration is described by equation (17) with $\sigma_p = 3.5$ arc sec and that launch shock errors are described by Eq. (46) and have a standard deviation of 1. arc min. Thus, the model misalignments themselves have the form

$$\theta_i = \mathbf{a}_i + \mathbf{b} \quad , \tag{78}$$

where

$$\mathbf{a}_i \sim \mathcal{N}(\mathbf{0}, (\sigma_p^2 + q)I_{3\times3}) \quad , \quad \mathbf{b} \sim \mathcal{N}(\mathbf{0}, \sigma_p^2 I_{3\times3}) \quad , \tag{79}$$

and were sampled accordingly. The nominal alignments, expressed in terms of the Gibbs vector [8], were taken to be

$$\mathbf{g}_1 = 0 \quad , \quad \mathbf{g}_2 = (2.5, 0, 0)^T \quad , \quad \mathbf{g}_3 = (0, 2.5, 0)^T \quad , \tag{80}$$

which is a typical set of alignments if sensor 1 is a Sun sensor and sensors 2 and 3 are star trackers.

One hundred samples of simulated data were generated. Table 2 shows the comparison of the model values and the corresponding estimates. Note that two-thirds of the estimates fall within one standard deviation of the model misalignments as expected.

Model Misalignments	Estimated Misalignments
37. arc sec	27. ± 29. arc sec
−23.	31. ± 29.
−58.	−72. ± 30.
−36.	−46. ± 29.
−63.	−18. ± 29.
4.	3. ± 29.
22.	18. ± 29.
−66.	−14. ± 29.
73.	69. ± 29.

Table 2
Comparison of Model and Estimated Misalignments

The estimated launch-shock standard deviation and its error from equations (67) and (69) was

$$(q^*)^{1/2} = 50. \pm 28. \text{ arc sec} \quad , \tag{81}$$

in good agreement with the model value of 1. arc min.

The level of agreement will be more manifest if we compare instead the components of Θ along the eigenvectors of $P_{\Theta\Theta}(+)$. If we define the orthogonal matrix C according to

$$C\, P_{\Theta\Theta}(+)C^T = P^D_{\Theta\Theta}(+) = \text{diag}(p_1, p_2, \ldots, p_{3n}) \quad , \tag{82}$$

and

$$\mathbf{\Phi} = C\,\Theta \quad , \tag{83}$$

then the estimates of the components of $\mathbf{\Phi}$ are uncorrelated and their estimate-error variances are given by the p_i, $i = 1, \ldots, 3n$. We call the components of $\mathbf{\Phi}$ the *eigenmisalignments.* If we compare these with the corresponding model values we find the result in Table 3.

Model Eigenmisalignments	Estimated Eigenmisalignments
60. arc sec	0. ± 50. arc sec
−18.	0. ± 50.
63.	0. ± 50.
33.	43. ± 10.
85.	90. ± 9.
−8.	−4. ± 5.
58.	57. ± 1.
−31.	−32. ± 1.
−17.	−17. ± 1.

Table 3
Comparison of Model and Estimated Eigenmisalignments

This shows the true level of agreement. Note that three of the estimates of the eigenmisalignments are exactly zero, a consequence of equation (17) above (i.e., that the prelaunch alignment estimation errors are identically distributed (but *not* independent)), and that the uncertainties in these estimates is given by the launch-shock error levels. This result is discussed in more detail in [4]. Also, three of the remaining estimates of the eigenmisalignments are nearly an order of magnitude more accurate than the other remaining three, a phenomenon which will also be discussed in detail in [4] and [9].

In the current example the three unobservable eigenmisalignments are quite large, on the order of $q^{1/2}$. It is these three eigenmisalignments which dominate the differences between the model values and the estimates in Table 2. Because these particular three eigenmisalignments contribute to the actual misalignments with equal coefficients, the standard deviations of the actual misalignment estimates will be roughly the same and the most important correlations will be on the order of $1/\sqrt{n}$. For the present example, in fact, had we chosen the three sensor boresights to be mutually orthogonal, we would have found nine of the correlations to be very nearly unity.

DISCUSSION AND CONCLUSIONS

We have presented a complete end-to-end methodology for estimating spacecraft sensor misalignments from prelaunch and inflight data which takes correct account of the prelaunch alignment calibration and treats the statistics of the inflight data correctly. As a necessary part of this program, we have developed a realistic representation of the prelaunch alignment estimate-error covariance and the errors due to launch shock and given a way to estimate the launch-shock error levels from the inflight data. The algorithms presented are fairly simple, owing this simplicity to the manner in which the misalignments have been defined. The performance of these algorithms has been illustrated with realistically simulated data. The methodology presented here represents a significant advance over one previously developed [10].

The estimate is not optimal over all $2n$ sensor measurements since effectively three of the measurements have been removed in order to achieve attitude independence. Thus, the results of this algorithm will differ somewhat from the results that would obtain, say, from a complete Kalman filter treating both the attitude and the alignments. If we assume that the only systematic error source for the attitude is the misalignments themselves, then those combinations of misalignments to which the set of derived measurements are sensitive will clearly not be very sensitive to the random attitude errors once sufficient data has been processed since these will tend to cancel one another. Likewise, those combinations of misalignments to which the derived measurements are not sensitive will be determined largely by the prelaunch calibration anyway. Thus, we expect little accuracy to be lost in our approximation compared to the Kalman filter. If the data are not simultaneous, however, or cannot be made simultaneous, then the methodology developed here will not be applicable and Kalman filter techniques will then need to be used [11].

There are two approaches to minimizing the effect of these unobservable errors. One approach is to mount one attitude sensor on a rigid plate on which the satellite payload is also mounted. In this way differences in geometrical distortion between that sensor and the payload are minimized. One then estimates only the relative misalignments with respect to the sensor collocated with the payload. This was the approach used for the Solar Maximum Mission (SMM), where the alignment cube for the fine pointing Sun sensor (FPSS) served also as the primary reference cube. However, the apparent misalignment of the sensor (and the payload) is not due to geometric distortion alone so that significant errors may exist in the supposed attitude of the payload. The contribution of the prelaunch alignment calibration, as degraded by launch-shock, can then be discarded without loss of accuracy.

A second approach is to use the fact that the payload itself is sensitive to the attitude. This was the situation on Magsat where the fine vector magnetometer, which was the principal scientific payload, was sensitive to the attitude in the same way as the attitude sensors. Thus, when estimating the harmonic expansion coefficients of the geomagnetic field the fine vector magnetometer misalignments could be estimated as well [12]. A detailed discussion of the Magsat experience will be given in [4].

The methodology developed here has been extended and applied to the estimation of the temperature dependence of the alignments for the Solar Maximum Mission [13].

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