International Symposium on Space Mechanics Toulouse, France, November 6–10, 1989

Fads and Fallacies in Spacecraft Sensor Alignment Estimation

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Abstract

Simple models are exploited to examine the qualitative properties of spacecraft sensor alignment estimation. A number of simple truths generally believed by workers are shown to be not quite correct.

Introduction

A number of incorrect beliefs are held about spacecraft sensor alignment estimation. These beliefs have influenced a number of missions. In some cases the errors introduced have been corrected in later processing. In many cases, however, their effect has gone unnoticed (because they are undetectable) and better attitude accuracies have been reported than have really been the case.

There exists a good deal of false mythology associated with alignment estimation. For reasons which remain unclear, there persists a misunderstanding about whether one should compute misalignments for all the sensors taking advantage of the prelaunch alignments to remove the ambiguity arising from an innate observability problem or whether one should estimate from the inflight data alone only relative alignments to one sensor, whose alignment is presumed to be unaffected by launch shock, and use this "alignment reference" to determine the full set of alignments. Clearly, the former method must yield more accurate results because it uses more information, but given the uncertainties surrounding launch shock it may not be clear how great the differences should be. A careful analysis of this problem indicates that the alignment estimates are decidedly better if the prelaunch calibration is incorporated in the inflight processing. This can be illustrated in a simple model.

Common belief has been that for sensors with narrow fields of view the misalignment about the boresights are unobservable. Careful examination shows that belief to be not quite correct.

The effect of unobservable launch-shock errors on attitude errors has never been explored. It turns out that these unobservable launch-shock errors lead to large, persistent, and unobservable attitude errors. These are investigated within the framework of the QUEST algorithm[1], which treats the vector observations in a manner similar to the alignment estimation algorithms developed earlier.

Basic Results

In an earlier work[2] we defined the alignment matrix S_i according to

$$\hat{\mathbf{W}}_{i,k} = S_i\,\hat{\mathbf{U}}_{i,k} \quad , \tag{1}$$

where $\hat{\mathbf{U}}_{i,k}$ is the direction measured by sensor i, $i = 1, \ldots, n$, at time t_k, and $\hat{\mathbf{W}}_{i,k}$ is the representation of the same unit vector in spacecraft body coordinates. S_i is the alignment matrix for sensor i and is necessarily proper orthogonal. We define further

$$S_i = M_i\, S_i^o \quad , \tag{2}$$

where S_i^o is the alignment matrix obtained from the prelaunch alignment calibration, and M_i is the misalignment matrix, a small rotation connecting S_i and S_i^o, which we may write approximately as

$$M_i \simeq I + \begin{bmatrix} 0 & \theta_{i3} & -\theta_{i2} \\ -\theta_{i3} & 0 & \theta_{i1} \\ \theta_{i2} & -\theta_{i1} & 0 \end{bmatrix} \quad , \tag{3}$$

where $\boldsymbol{\theta}_i \equiv [\theta_{i1}, \theta_{i2}, \theta_{i3}]^T$ is the misalignment vector for sensor i. The inflight alignment calibration determines $\boldsymbol{\theta}_i$, $i = 1, \ldots, n$.

If we define an "uncalibrated" body-referenced observation vector, $\hat{\mathbf{W}}^o_{i,k}$, according to

$$\hat{\mathbf{W}}^o_{i,k} \equiv S_i^o \, \hat{\mathbf{U}}_{i,k} \quad , \tag{4}$$

then

$$\hat{\mathbf{W}}^o_{i,k} = M_i^T A_k \hat{\mathbf{V}}_{i,k} + \Delta\hat{\mathbf{W}}^o_{i,k} \quad , \tag{5}$$

where $\hat{\mathbf{V}}_{i,k}$ is the representation of the unit vector in the primary reference frame (i.e., the frame to which the attitude is referred) and A_k is the attitude matrix. The measurement noise $\Delta\hat{\mathbf{W}}^o_{i,k}$, is assumed to be white and uncorrelated between sensors. We see immediately from Eq. (5) that a common rotation in the misalignment matrices is indistinguishable from the opposite rotation in the attitude matrix. Therefore, not all misalignments are observable from inflight data alone, a fact which does not seem to have always been understood[3].

A set of pseudo-measurements can be constructed which depend only on the misalignments and not on the attitude according to

$$\begin{aligned} z_{ij,k} &\equiv \hat{\mathbf{W}}^o_{i,k} \cdot \hat{\mathbf{W}}^o_{j,k} - \hat{\mathbf{V}}_{i,k} \cdot \hat{\mathbf{V}}_{j,k} \quad , \\ &\simeq (\hat{\mathbf{W}}^o_{i,k} \times \hat{\mathbf{W}}^o_{j,k}) \cdot (\boldsymbol{\theta}_i - \boldsymbol{\theta}_j) + \Delta z_{ij,k} \quad , \end{aligned} \tag{6}$$

where terms which are higher order in the misalignments have been discarded. The measurement noise is given to lowest order by

$$\Delta z_{ij,k} \simeq \hat{\mathbf{W}}^o_{i,k} \cdot \Delta\hat{\mathbf{W}}^o_{j,k} + \hat{\mathbf{W}}^o_{j,k} \cdot \Delta\hat{\mathbf{W}}^o_{i,k} \quad . \tag{7}$$

We note that there are $n(n-1)/2$ pseudo-measurements $z_{ij,k}$ with $i < j$ although from the above discussion there can be at most $2n-3$ attitude-independent measurements at each time t_k.

Thus, only a subset of these $n(n-1)/2$ pseudo-measurements can be statistically independent. A suitable subset is given by[2]

$$\mathbf{Z}_k \equiv [z_{12,k}, \ldots, z_{1n.k}, z_{23,k}, \ldots, z_{3n,k}]^T \quad , \tag{8}$$

provided that $\hat{\mathbf{W}}^o_{1,k}$ and $\hat{\mathbf{W}}^o_{2,k}$ are not mutually parallel or parallel to any of the remaining unit-vector measurements. Thus, we write,

$$\mathbf{Z}_k = H_k \, \boldsymbol{\Theta} + \Delta\mathbf{Z}_k \quad , \tag{8}$$

where

$$\boldsymbol{\Theta} \equiv [\boldsymbol{\theta}_1^T, \ldots, \boldsymbol{\theta}_n^T]^T \quad , \tag{9}$$

is the total alignment vector and has dimension $3n$. Note that H_k cannot be full rank.

If $\boldsymbol{\Theta}^*(-)$ is the estimate of the total alignment vector based on the prelaunch calibration (hence, $\mathbf{0}$), and $P_{\Theta\Theta}(-)$ is the associated estimate error covariance matrix, then the inflight *a posteriori* maximum likelihood estimate and covariance matrix are given by[2]

$$P_{\Theta\Theta}^{-1}(+) \, \boldsymbol{\Theta}^*(+) = \sum_{k=1}^{N} H_k^T \, P_{\mathbf{Z}_k}^{-1} \, \mathbf{Z}_k \quad , \tag{10}$$

$$P_{\Theta\Theta}^{-1}(+) = P_{\Theta\Theta}^{-1}(-) + \sum_{k=1}^{N} H_k^T \, P_{\mathbf{Z}_k}^{-1} \, H_k \quad . \tag{11}$$

These relationships are the basis for the present study.

Note that three linear combinations of the components of $\boldsymbol{\Theta}^*(+)$ are given by the prelaunch values (i.e., 0). For this reason one has often estimated instead the relative alignment vector defined by

$$\boldsymbol{\Theta}^{\text{rel}} \equiv [\boldsymbol{\theta}_2^T - \boldsymbol{\theta}_1^T, \, \ldots \, , \boldsymbol{\theta}_n^T - \boldsymbol{\theta}_1^T]^T \quad , \tag{12}$$

a vector of dimension $3n - 3$.

Relative versus Absolute Alignment Estimation

It is part of the mythology of alignment estimation that it is more accurate to estimate the relative alignments (because they are completely observable from inflight data alone) than the absolute alignments. Some reports[4] even bolster this claim by presenting simulation results which demonstrate (quite correctly) that the variances of the relative alignments are smaller than those of the absolute misalignments. Unfortunately, when we come to estimate the attitude, it is the absolute alignments which we require in order to transform data from the sensor frame to the body frame. Thus, if only the relative alignments are estimated, some assumption must be made about the value of the absolute alignment of one of the sensors. It has been the practice in these cases to set $\boldsymbol{\theta}_1^*(+) \equiv \mathbf{0}$, which makes sense only if there is strong reason to believe *a priori* that the mislignment of sensor 1 relative to the spacecraft payload will be much smaller than the misalignments of the other sensors. This is not always the case. In addition, works which estimate relative alignments only have also tended to discard the prelaunch alignment calibration information. Thus, users of this naive approach make two serious approximations.

Any difference in the accuracies claimed for those two methods can result only from the additional approximations made in the former. However, unrealistically higher accuracies are sometimes claimed[4] for the "relative alignments" because the correct statistics of $\boldsymbol{\theta}_1^*$ are not taken into account in the covariance analysis. We will examine the actual effect of these approximations in a simple but realistic model and compare the results of this naive approach with the more correct and statistically consistent maximum likelihood estimates.

The inflight data can be represented by a single effective measurement, $\mathbf{Z}$, of the form

$$\mathbf{Z} = H \, \boldsymbol{\Theta} + \mathbf{v} \quad , \tag{13}$$

with

$$\mathbf{v} \sim \mathcal{N}(\mathbf{0}, \; R) \quad . \tag{14}$$

The prior-free estimate of the relative misalignments, for example, is just such an effective measurement. Such an effective measurement can always be generated from

$$\mathbf{Z} = \sum_{k=1}^{N} H_k^T \, P_{\mathbf{Z}_k}^{-1} \, \mathbf{Z}_k \quad , \tag{15}$$

for which

$$R^{-1} = \sum_{k=1}^{N} H_k^T \, P_{\mathbf{Z}_k}^{-1} \, H_k \quad . \tag{16}$$

$\mathbf{Z}$ is a sufficient statistic[5] for the misalignments. Note that for this choice of the sufficient statistic both H and R are singular. This need not always be true for every sufficient statistic (however, $H^T R^{-1} H$ must be singular since the full set of misalignments cannot be determined from the inflight data alone). One achieves a non-singular R simply by eliminating the three redundant components of Z.

Equations (15) and (16) entail no approximation. In terms of this sufficient statistic, the correctly computed maximum likelihood estimate of the misalignments will, therefore, have *a posteriori* covariance

$$P_{\Theta\Theta}(+) = \left[P_{\Theta\Theta}^{-1}(-) + H^T R^{-1} H\right]^{-1} \quad . \tag{17}$$

The naive intuitive approach to relative misalignment estimation sets

$$\boldsymbol{\theta}_1^{*\,\mathrm{naive}}(+) \equiv \mathbf{0} \quad , \tag{18}$$

and estimates $\boldsymbol{\Theta}' \equiv [\boldsymbol{\theta}_2^T, \ldots, \boldsymbol{\theta}_n^T]^T$ by minimizing

$$J^{\mathrm{rel}} \equiv \frac{1}{2}\left(\mathbf{Z} - H'\,\boldsymbol{\Theta}'\right)^T R^{-1}\left(\mathbf{Z} - H'\,\boldsymbol{\Theta}'\right) \quad , \tag{19}$$

where

$$H \equiv [\,h_1 \ |\, H'\,] \quad . \tag{20}$$

Thus, this relative misalignment estimate is equivalent to the prior-free relative misalignment estimate but wrongly interpreted as being the absolute misalignment vector. The naive estimate of $\boldsymbol{\Theta}'$ is, therefore,

$$\boldsymbol{\Theta}'^{\,*\,\mathrm{naive}} = \left(H'^T R^{-1} H'\right)^{-1} H'^T R^{-1} \mathbf{Z} \quad , \tag{21}$$

supplemented by Eq. (18).

From Eqs. (13), (18), and (21) it follows for the complete misalignment vector that

$$\boldsymbol{\Theta}^{*\,\mathrm{naive}}(+) = \boldsymbol{\Theta} + G_{\theta_1}\,\boldsymbol{\theta}_1 + G_v\,\mathbf{v} \quad , \tag{22}$$

where

$$G_{\theta_1} = \begin{bmatrix} -I_{3\times 3} \\ \left(H'^T R^{-1} H'\right)^{-1}\left(H'^T R^{-1} h_1\right) \end{bmatrix} \quad , \tag{23}$$

$$G_v = \begin{bmatrix} 0_{3\times 3} \\ \left(H'^T R^{-1} H'\right)^{-1} H'^T R^{-1} \end{bmatrix} \quad . \tag{24}$$

The naive relative alignment estimates are seen to be biased by terms linear in the true value of $\boldsymbol{\theta}_1$, which is not surprising.

The true covariance matrix of the naive relative alignment estimates is thus

$$P_{\Theta\Theta}^{\mathrm{naive}} = G_{\theta_1}\,P_{\theta_1\theta_1}(-)\,G_{\theta_1}^T + G_v\,R\,G_v^T \quad , \tag{25}$$

while the covariance which is incorrectly claimed by the practitioners of this naive method is

$$\text{“}P_{\Theta'\Theta'}^{\mathrm{naive}}(+)\text{”} = \left(H'^T R^{-1} H'\right)^{-1} \quad . \tag{26}$$

The quotation marks remind us that the covariance is based on incorrect statistical assumptions.

We can evaluate all three expressions in a common model. We assume for the sake of simplicity that

$$R = \sigma^2\, I_{3n\times 3n} \quad , \tag{27}$$

and

$$H = I_{3n\times 3n} - \frac{1}{n}\,L_{3n\times 3n} \quad , \tag{28}$$

where

$$L_{3m\times 3n} \equiv \begin{bmatrix} I & I & \cdots & I \\ I & I & \cdots & I \\ \vdots & \vdots & \ddots & \vdots \\ I & I & \cdots & I \end{bmatrix} \quad , \tag{29}$$

and $L_{3m\times 3n}$ has $3m$ rows and $3n$ columns. This is the simplest model for $\mathbf{Z}$ which is a function of the relative alignments alone as required by Eq. (6). (Note that this $\mathbf{Z}$ has only $(3n-3)$ statistically independent components, as required.) Using this model and taking the *a priori* inflight covariance matrix to be given by

$$P_{\Theta\Theta}(-) = \sigma_o^2 \, I_{3n\times 3n} \tag{30}$$

leads to

$$P_{\Theta\Theta}(+) = \left(\sigma^{-2} + \sigma_o^{-2}\right)^{-1} \left(I_{3n\times 3n} - \frac{1}{n} L_{3n\times 3n} \right) + \sigma_o^2 \, \frac{1}{n} \, L_{3n\times 3n} \quad , \tag{31}$$

$$\text{“}P_{\Theta\Theta}^{\text{naive}}(+)\text{”} = \begin{bmatrix} 0_{3\times 3} & 0 \\ 0 & \sigma^2 (I_{3(n-1)\times 3(n-1)} + L_{3(n-1)\times 3(n-1)}) \end{bmatrix} \quad , \tag{32}$$

and

$$P_{\Theta\Theta}^{\text{naive}} = \text{“}P_{\Theta\Theta}^{\text{naive}}(+)\text{”} + \sigma_o^2 \, L_{3n\times 3n} \quad . \tag{33}$$

Equation (33) states that the true covariance for the naive relative alignment estimation procedure is equal to the one claimed for that method based on its incorrect statistical assumptions plus a correction term. Note that the correction term is linear in σ_o^2, which is large due to large launch-shock errors.

We can compare the three covariances by computing the average variances in each case (defined as $1/(3n)$ times the trace of the covariance matrix). The result for the true average variance of the correctly computed maximum likelihood estimate (from Eq. (31)) is

$$\langle \sigma_{\text{true}}^2 \rangle = (1/n)\, \sigma_o^2 + (1 - (1/n)) \left((\sigma^{-2} + \sigma_o^{-2} \right)^{-1} \quad . \tag{34}$$

The average variance claimed by the naive relative alignment estimation based on its own false statistical assumptions is

$$\langle \text{“}\sigma_{\text{naive}}^2\text{”} \rangle = 2\,(1 - (1/n))\, \sigma^2 \quad . \tag{35}$$

Finally, the true typical variance for the naive relative estimates is

$$\langle \sigma_{\text{naive}}^2 \rangle = \sigma_o^2 + 2\,(1 - (1/n))\, \sigma^2 \quad . \tag{36}$$

Since the *a priori* covariance is the largest contributor to each of these expressions, the naive relative estimates clearly are the poorer result (by a factor n), although based on the incorrect statistical assumptions on which the naive estimators are based, they would seem (without closer scrutiny) to be the best.

Physically what is happening is that by setting $\boldsymbol{\theta}_1^*(+) \equiv \mathbf{0}$ in the naive approach, the entire prelaunch uncertainty is forced into that quantity, and each of the other misalignment vectors is shifted in the opposite sense by the same amount. The more consistent maximum likelihood estimate, which does not prejudice the estimation against one misalignment, spreads this uncertainty over all the misalignments and effectively reduces their effect by a factor $1/\sqrt{n}$. An eigenvalue analysis of the two covariances (calculated with realistic statistics) show that both have $(3n-3)$ of their eigenvalues equal roughly to σ^2, as we would expect intuitively, since $(3n-3)$ misalignments should be determined accurately by the inflight data no matter almost what crimes are committed in constructing the estimators. For the consistent maximum likelihood approach the remaining three eigenvalues are σ_o^2, while for the naive approach they are approximately $n\sigma_o^2$, which is considerably larger.

Alignment Estimation Accuracies for Narrow Fields of View

When the field of view of the sensor is small it becomes difficult to distinguish misalignments about the sensor boresights from those about the other sensor axes. Common usage has been to simply restrict allowable misalignments to the axes normal to the boresight. We will investigate the necessity and wisdom of such a procedure within a simple but realistic model.

Suppose that the spacecraft is equipped with three vector sensors, each with a limited field of view and with boresights nominally along the three spacecraft body axes. We will assume that each frame contains measurements for all three sensors and that the distribution of these measurements about each sensor boresight is axially symmetric with a root-mean-square angular radius of $\sqrt{2}\alpha$ (i.e., the root-mean-square spread of each of the components of $\hat{\mathbf{W}}_{i,k}$ about the boresight is α). Thus, we may write the measurement equation as

$$\mathbf{Z}_k = \begin{bmatrix} z_{23,k} \\ z_{31,k} \\ z_{12,k} \end{bmatrix} = H_k\,\mathbf{\Theta} + \Delta\mathbf{Z}_k \quad , \tag{37}$$

where the reordering of the components and sign changes will serve to give the measurement vector a cyclic symmetry and simplify later calculations. The sensitivity matrix, H_k, is given now by

$$H_k = \begin{bmatrix} \mathbf{0}^T & (\hat{\mathbf{W}}^o_{2,k} \times \hat{\mathbf{W}}^o_{3,k})^T & -(\hat{\mathbf{W}}^o_{2,k} \times \hat{\mathbf{W}}^o_{3,k})^T \\ -(\hat{\mathbf{W}}^o_{3,k} \times \hat{\mathbf{W}}^o_{1,k})^T & \mathbf{0}^T & (\hat{\mathbf{W}}^o_{3,k} \times \hat{\mathbf{W}}^o_{1,k})^T \\ (\hat{\mathbf{W}}^o_{1,k} \times \hat{\mathbf{W}}^o_{2,k})^T & -(\hat{\mathbf{W}}^o_{1,k} \times \hat{\mathbf{W}}^o_{2,k})^T & \mathbf{0}^T \end{bmatrix} \quad . \tag{38}$$

We will assume that

$$\Delta\mathbf{Z}_k \sim \mathcal{N}(\mathbf{0},\, P_{\mathbf{Z}_k}) \quad , \tag{39}$$

with

$$P_{\mathbf{Z}_k} = \sigma^2\, I_{3\times 3} \quad . \tag{40}$$

Note from Eq. (38) that due to the narrow fields of view

$$\mathbf{Z}_k \simeq \begin{bmatrix} \hat{\mathbf{e}}_1 \cdot (\boldsymbol{\theta}_2 - \boldsymbol{\theta}_3) \\ \hat{\mathbf{e}}_2 \cdot (\boldsymbol{\theta}_3 - \boldsymbol{\theta}_1) \\ \hat{\mathbf{e}}_3 \cdot (\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2) \end{bmatrix} + \Delta\mathbf{Z}_k \quad , \tag{41}$$

so that only $\theta_{2x} - \theta_{3x}$, $\theta_{3y} - \theta_{1y}$, and $\theta_{1z} - \theta_{2z}$ will be determined with high accuracy. Three combinations of the misalignments will be determined not at all by the inflight data, and the remaining three will be determined poorly. For general n (and not very undesirable geometry) we note that Eq. (41) will have $(2n-3)$ statistically independent components. Thus, we expect that $(2n-3)$ combinations will be determined well from the inflight data, n combinations relatively poorly, and 3 combinations not at all. The loss in alignment estimation accuracy might seem, therefore, to be close to our naive expectations.

To appreciate the magnitudes involved let us compute the estimate error covariance matrix in some detail. From Eq. (40) the Fisher information matrix for the inflight data is

$$\begin{aligned} P^{-1}_{\Theta\Theta}(\text{inflight}) &= \sum_{k=1}^{N} H_k^T\, P^{-1}_{\mathbf{Z}_k}\, H_k \\ &= \sigma^{-2} \sum_{k=1}^{N} H_k^T\, H_k \quad , \end{aligned} \tag{42}$$

and for N very large,

$$P_{\Theta\Theta}^{-1}(\text{inflight}) = N\,\sigma^{-2}\,\left\langle H_k^T\,H_k \right\rangle \quad , \tag{43}$$

where $\langle\,\cdot\,\rangle$ denotes an average over the orientation of the observation within the field of view. Evaluating this average leads to

$$P_{\Theta\Theta}^{-1}(\text{inflight}) = N\,\sigma^{-2} \times$$

$$\begin{bmatrix} 2\beta & 0 & 0 & -\beta & 0 & 0 & -\beta & 0 & 0 \\ 0 & 1+\beta & 0 & 0 & -\beta & 0 & 0 & -1 & 0 \\ 0 & 0 & 1+\beta & 0 & 0 & -1 & 0 & 0 & -\beta \\ -\beta & 0 & 0 & 1+\beta & 0 & 0 & -1 & 0 & 0 \\ 0 & -\beta & 0 & 0 & 2\beta & 0 & 0 & -\beta & 0 \\ 0 & 0 & -1 & 0 & 0 & 1+\beta & 0 & 0 & -\beta \\ -\beta & 0 & 0 & -1 & 0 & 0 & 1+\beta & 0 & 0 \\ 0 & -1 & 0 & 0 & -\beta & 0 & 0 & 1+\beta & 0 \\ 0 & 0 & -\beta & 0 & 0 & -\beta & 0 & 0 & 2\beta \end{bmatrix} \quad , \tag{44}$$

with

$$\beta \equiv \alpha^2 \quad . \tag{45}$$

This matrix can be simplified by defining a new total misalignment vector, $\mathbf{\Phi}$, by

$$\begin{aligned} \mathbf{\Phi} &\equiv [\,\Theta_1,\,\Theta_4,\,\Theta_7,\,\Theta_5,\,\Theta_8,\,\Theta_2,\,\Theta_9,\,\Theta_3,\,\Theta_6,\,]^T \\ &\equiv T\,\mathbf{\Theta} \quad . \end{aligned} \tag{46}$$

Since $\mathbf{\Phi}$ is simply a reordering of the components of $\mathbf{\Theta}$, it follows that T is orthogonal. In terms of $\mathbf{\Phi}$

$$P_{\Phi\Phi}^{-1}(\text{inflight}) = T\,P_{\Theta\Theta}^{-1}(\text{inflight})\,T^T = N\,\sigma^2 \begin{bmatrix} \mathcal{M} & O_{3\times3} & O_{3\times3} \\ O_{3\times3} & \mathcal{M} & O_{3\times3} \\ O_{3\times3} & O_{3\times3} & \mathcal{M} \end{bmatrix} \quad , \tag{47}$$

with

$$\mathcal{M} = \begin{bmatrix} 2\beta & -\beta & -\beta \\ -\beta & 1+\beta & -1 \\ -\beta & -1 & 1+\beta \end{bmatrix} \quad . \tag{48}$$

Thus, the eigenvalues of $P_{\Phi\Phi}^{-1}(\text{inflight})$ (and, therefore, of $P_{\Theta\Theta}^{-1}(\text{inflight})$) each have a three-fold degeneracy. The eigenvalues of $\mathcal{M}$ are simply

$$\lambda_1 = 0 \quad , \quad \lambda_2 = 3\beta \quad , \quad \lambda_3 = 2+\beta \quad . \tag{49}$$

The vanishing of one of the eigenvalues of $\mathcal{M}$ is required by our earlier discussion.

If we assume as before that the *a priori* covariance matrix for the misalignments to be

$$P_{\Theta\Theta}^{-1}(-) = \sigma_o^2\,I_{9\times9} \quad , \tag{50}$$

then the three eigenvalues of $P_{\Theta\Theta}^{-1}(+)$ are

$$\sigma_1^2 = \sigma_o^2 \quad , \tag{51a}$$

$$\sigma_2^2 = \left(\frac{1}{\sigma_o^2} + \frac{3N\alpha^2}{\sigma^2}\right)^{-1} \quad , \tag{51b}$$

$$\sigma_3^2 = \left(\frac{1}{\sigma_o^2} + \frac{N(2+\alpha^2)}{\sigma^2}\right)^{-1} \quad . \tag{51c}$$

Choosing values

$$\sigma_o = 1. \text{ arc min} \quad , \quad \sigma = 10. \text{ arc sec} \quad , \quad \alpha = 5. \text{ deg} \quad , \quad N = 100 \quad , \tag{52}$$

the three standard deviations become roughly

$$\sigma_1 = 1. \text{ arc min} \quad , \quad \sigma_2 = 13. \text{ arc sec} \quad , \quad \sigma_3 = 0.7 \text{ arc sec} \quad . \tag{53}$$

The restricted field of view is seen to be not a serious impediment to estimating alignments accurately although the differences in alignment accuracies are substantial. The most serious deficiency, of course, is the complete lack of observability from inflight data of three combinations of the misalignments.

Note, however, that the three misalignment vectors which are poorly determined inflight, i.e., the three eigenvectors of $P_{\Theta\Theta}^{-1}$(inflight) with eigenvalue 3β, are given by

$$\frac{1}{\sqrt{6}}\begin{bmatrix} 2 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} \quad , \quad \frac{1}{\sqrt{6}}\begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 2 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} \quad , \quad \frac{1}{\sqrt{6}}\begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 2 \end{bmatrix} \quad .$$

These are not the three boresight vectors. The loss in alignment estimation accuracy due to restricted fields of view is, therefore, somewhat different from not being able to estimate well the alignments about the n sensor boresights.

Influence of Misalignment Accuracy on Attitude Accuracy

The error in the misalignment estimates necessarily translates into attitude error. To determine the degree to which this occurs we suppose that attitude will be determined using the QUEST algorithm[1]. Thus, we assume that the attitude algorithm will only use the estimates of the final inflight alignment calibration to correct the assumed alignments but not try to work the detailed covariance matrix of the inflight alignment estimates into the attitude estimator (except perhaps for adjusting the values of the variances which are intrinsic to the QUEST algorithm). This is probably a reasonable, if suboptimal, approach.

Let $\Delta\boldsymbol{\theta}_i$, as usual, denote the misalignment vector estimate error for sensor i, and let $\boldsymbol{\xi}_k^\theta$ be the additional attitude error arising from the misalignment errors. In the absence of misalignment errors, the optimal QUEST attitude matrix, A_k^* , minimizes[1]

$$J_k \equiv \frac{1}{2}\sum_{i=1}^{n} \frac{1}{\sigma_{i,k}^2}\,|\hat{\mathbf{W}}_{i,k} - A_k\,\hat{\mathbf{V}}_{i,k}|^2 \quad . \tag{54}$$

In the presence of alignment errors, the correction, $\boldsymbol{\xi}_k^\theta$, to the QUEST attitude (without correcting the weights for these alignment errors) minimizes

$$J_k' \equiv \frac{1}{2}\sum_{i=1}^{n} \frac{1}{\sigma_{i,k}^2}\left| e^{[[\,\Delta\boldsymbol{\theta}_i\,]]}\hat{\mathbf{W}}_{i,k} - e^{[[\,\boldsymbol{\xi}_k^\theta\,]]}A_k^*\,\hat{\mathbf{V}}_{i,k}\right|^2 \quad , \tag{55}$$

$$= \frac{1}{2}\sum_{i=1}^{n} \frac{1}{\sigma_{i,k}^2}\left| \hat{\mathbf{W}}_{i,k} - e^{[[\,\boldsymbol{\xi}_k^\theta - \Delta\boldsymbol{\theta}_i\,]]}A_k^*\,\hat{\mathbf{V}}_{i,k}\right|^2 \quad , \tag{56}$$

$$= \frac{1}{2}\sum_{i=1}^{n} \frac{1}{\sigma_{i,k}^2}\left| \hat{\mathbf{W}}_{i,k} - A_k^*\,\hat{\mathbf{V}}_{i,k} + [[\,A_k^*\,\hat{\mathbf{V}}_{i,k}\,]]\,(\boldsymbol{\xi}_k^\theta - \Delta\boldsymbol{\theta}_i)\right|^2 \quad , \tag{57}$$

and the equalities are true to $O(|\boldsymbol{\xi}_k^\theta| + |\Delta\boldsymbol{\theta}_i|)$. Minimizing J'_k over $\boldsymbol{\xi}_k^\theta$ leads to

$$\boldsymbol{\xi}_k^{\theta\,*} = F_k^{-1}\left(\sum_{i-1}^{n} F_{i,k}\,\Delta\boldsymbol{\theta}_i + \sum_{i=1}^{n}\frac{1}{\sigma_{i,k}^2}\,[[\,A_k^*\,\hat{\mathbf{V}}_{i,k}\,]]\,\hat{\mathbf{W}}_{i,k}\right) \quad , \tag{58}$$

where

$$F_{i,k} = \frac{1}{\sigma_{i,k}^2}\left[I_{3\times3} - \left(A_k^*\,\hat{\mathbf{V}}_{i,k}\right)\left(A_k^*\,\hat{\mathbf{V}}_{i,k}\right)^T\right] \quad , \tag{59}$$

which is the Fisher information matrix[5] for the attitude associated with the single measurement $\hat{\mathbf{W}}_{i,k}$, and

$$F_k = \sum_{i=1}^{n} F_{i,k} \quad , \tag{60}$$

which is the Fisher information matrix for the attitude arising from all the measurements at time t_k.

By definition, $\boldsymbol{\xi}_k^\theta$ must vanish when the $\Delta\boldsymbol{\theta}_i$ all vanish. Therefore, the second term in Eq. (58) must vanish to first order, leading finally to

$$\boldsymbol{\xi}_k^{\theta\,*} = F_k^{-1}\sum_{i-1}^{n} F_{i,k}\,\Delta\boldsymbol{\theta}_i \quad . \tag{61}$$

The misalignment errors, thus, lead to a random bias in the attitude. The attitude errors arising from the random sensor noise have covariance F_k^{-1}. Thus, the total covariance of the QUEST attitude solutions taking account of both sensor noise and misalignment estimate errors is

$$P_{\xi_k\xi_{k'}} = \delta_{kk'}\,F_k^{-1} + F_k^{-1}\left(\sum_{i,j}^{n} F_{i,k}\,P_{\theta\theta}^{ij}(+)\,F_{j,k'}\right)F_{k'}^{-1} \quad , \tag{62}$$

where $P_{\theta\theta}^{ij}(+)$ is the appropriate submatrix of the misalignement estimate error covariance matrix as given by Eq. (11). Note that the attitudes are now autocorrelated due to the misalignment error.

How large are the two contributions to Eq. (62)? Consider the model of the last section, which assumed that the measurements are always close to the spacecraft body axes. Let us further assume that N, the number of frames of attitude data, is sufficiently large that the only important alignment errors come from the components of the misalignments that are unobservable from inflight data. Then

$$P_{\theta\theta}^{ij} = \frac{1}{3}\,\sigma_o^2\,I_{3\times3} \quad , \tag{63}$$

and

$$F_k^{-1} = \frac{1}{2}\,\sigma^2\,I_{3\times3} \quad . \tag{64}$$

Hence,

$$P_{\xi_k\xi_{k'}} = \left(\frac{1}{2}\,\sigma^2\,\delta_{kk'} + \frac{1}{3}\,\sigma_o^2\right)I_{3\times3} \quad . \tag{65}$$

For the values assumed in the previous example the random sensor measurements contribute 7. arc sec/axis to the attitude error, while the unobservable (i.e., from inflight data) misalignments contribute 34. arc sec/axis. Thus, the unobservable misalignments can be the largest contributor to the attitude errors.

Discussion and Conclusions

We have demonstrated a number of qualitative results on spacecraft sensor alignment estimation which do not seem to be generally known. For a system of n vector sensors three linear combinations of the misalignments are unobservable, and for sensors with narrow fields of view $2n - 3$ misalignments will be estimated well and the remaining n somewhat less well. These n misalignments, however, do not correspond exactly to the misalignments about the sensor boresights. Given sufficient data, however, it is only the three unobservable combinations of the misalignments which limit the accuracy of the inflight alignment estimation procedure. Since these three combinations are just the average of the misalignment vectors, they corrupt all other misalignments equally. Thus, if σ_o is the standard deviation of the post-launch alignments (i.e., corrupted by launch shock), the effect of the inflight calibration is to reduce this standard deviation to $\sigma_o/\sqrt{n}$. This is reflected in the examples. Additional accuracy is in general not possible unless the output of the payload also provides equivalent attitude information or it is known that the misalignment of one sensor from the payload is truly negligible. In many cases this is not the case.

References

[1]Shuster, M. D., and Oh, S. D., "Three-Axis Attitude Determination from Vector Observations," *Journal of Guidance and Control*, Vol. 4, No. 1, pp. 70–77, 1981.

[2]Bierman, G. J., and Shuster, M. D., "Spacecraft Alignment Estimation," *Proceedings, 27th IEEE Conference on Decision and Control*, Austin, Texas, 1988.

[3]des Jardins, R., "In-Orbit Startracker Misalignment Estimates on the OAO," *Proceedings, Symposium of Spacecraft Attitude Determination*, El Segundo, Calif., Sept. 30–Oct. 2, 1969, p.1.

[4]Fang, B., Gambardella, P., and McLaughlin, S., "Analysis of Alignment Algorithm for Space Telescope Fixed-Head Star Tracker and Fine Guidance Sensor," Computer Sciences Corporation, CSC/TM-84/6117, 1984.

[5]Sorenson, H. W., *Parameter Estimation.* Marcel Dekker, New York, 1980.