PAPER AAS 89-365

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AAS/AIAA Astrodynamics Specialist Conference

STOWE, VERMONT, AUGUST 7-10, 1989
AAS Publications Office, P.O. Box 28130, San Diego, CA 92128

Astrodynamics 1989
Advances in the Astronautical Sciences, Vol. 71

AAS 89-365

ATTITUDE DETERMINATION FOR THE STAR TRACKER MISSION

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An efficient Kalman filter for spacecraft attitude has been developed for the Star Tracker mission which makes special use of the presence of a CCD star camera. This device is capable in its current configuration of measuring the directions of five stars simultaneously. For such a device the prior-free maximum-likelihood estimate of the attitude based on these simultaneous measurements provides a sufficient statistic which greatly reduces the computational burden of the filter. On the basis of the outputs of this sensor alone, the on-board attitude determination system will determine both three-axis attitude and angular velocity using a dynamical model in which the spacecraft torque is modeled as a Gaussian white noise. Initial simulations of the attitude determination system performance show good performance of the filter.

INTRODUCTION

The Star Tracker experiment will demonstrate near real-time, autonomous satellite attitude determination, accurate to one arc second, using a state-of-the-art charge-coupled-device (CCD) star camera. The experiment, sponsored by the U.S. Army Engineer Topographic Laboratories, will fly on a NASA spacecraft which will be launched for a forty hour mission from the space shuttle and later retrieved for return to Earth. The new camera and accompanying attitude software represent a significant step toward spacecraft autonomy. Extensive simulation software and hardware have been developed for testing the experiment in different configurations during the project[1].

Because the Star Tracker spacecraft carries only a single precise attitude sensor, the CCD star camera, from which both attitude and angular velocity must be determined, it is evident that a Kalman filter[2] must be implemented. However, because the star camera furnishes simultaneous sightings of as many as five different stars in its 7.2 deg by 9. deg field of view, it is also possible to determine three-axis attitude at any sample time without the filter. Thus, it seemed advantageous to design a Kalman filter which would take as its input the attitude computed from one frame of data. The single-frame attitude thus becomes a sufficient statistic for the attitude and angular

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velocity which is used by the Kalman filter as an effective measurement. This approach has several advantages. Firstly, very efficient algorithms exist for computing this single-frame attitude[3]; and, secondly, the fact that attitude can be reliably calculated outside the filter provides a check on the filter performance. In simulations the filter has performed very well.

We begin this report by presenting the state model and the prediction algorithms of the filter. Since the spacecraft dynamics are not presented explicitly, the spacecraft torque is modeled simply as a Gaussian white noise process. The results here are straightforward and follow closely the methods developed in Ref. 2. Next we develop an effective measurement model based on the outputs of the QUEST attitude determination algorithm[3]. Since the QUEST algorithm is a maximum likelihood estimator and the Kalman filter is simply a sequential mechanization of the maximum likelihood estimate when the measurement and process noise are Gaussian, it follows that using the QUEST algorithm as a front end to the filter entails no approximation. A second innovation of the present work is the manner in which the filter is initialized. Generally, when there is no *a priori* information, the Kalman filter is initialized by using a very large initial covariance and any convenient initial condition, it being hoped that the filter will eventually converge to the statistically correct behavior. This brutal treatment is usually unnecessary since it is easy enough to determine a batch maximum likelihood estimate and estimate error covariance from the first two frames of data. Thus, the filter begins with the correct statistical parameters and can provide useful results immediately. The steps needed for the initialization of the filter are described in detail.

A number of simulations have been carried out to demonstrate the filter performance.

DYNAMICAL MODEL AND THE FILTER PREDICTION

The State Equation and State Prediction

The dynamical model for the Star Tracker attitude determination system is simply

$$\frac{d}{dt}\bar{q}(t) = \frac{1}{2}\,\Omega(\boldsymbol{\omega}(t))\,\bar{q}(t) \quad , \tag{1}$$

$$\frac{d}{dt}\boldsymbol{\omega}(t) = \boldsymbol{\eta}(t) \quad . \tag{2}$$

Here $\bar{q}(t)$ is the attitude quaternion and $\boldsymbol{\omega}(t)$ is the body-referenced angular velocity. $\Omega(\boldsymbol{\omega})$ is given by

$$\Omega(\boldsymbol{\omega}) \equiv \begin{bmatrix} 0 & \omega_3 & -\omega_2 & \omega_1 \\ -\omega_3 & 0 & \omega_1 & \omega_2 \\ \omega_2 & -\omega_1 & 0 & \omega_3 \\ -\omega_1 & -\omega_2 & -\omega_3 & 0 \end{bmatrix} \quad , \tag{3}$$

and $\boldsymbol{\eta}(t)$ is white Gaussian noise with power spectral density $Q(t)$. The spacecraft state vector is simply

$$\mathbf{x}(t) = \begin{bmatrix} \bar{q}(t) \\ \boldsymbol{\omega}(t) \end{bmatrix} \quad . \tag{4}$$

Following Ref. 2 the attitude error state is defined by*

$$\delta\mathbf{x}(t) = \begin{bmatrix} \Delta\boldsymbol{\theta}(t) \\ \Delta\boldsymbol{\omega}(t) \end{bmatrix} \tag{5}$$

where

$$\bar{q}(t) \equiv \frac{1}{\sqrt{1+|\Delta\boldsymbol{\theta}(t)|^2/4}} \begin{bmatrix} \Delta\boldsymbol{\theta}(t)/2 \\ 1 \end{bmatrix} \otimes \bar{q}^*(t) \quad , \tag{6}$$

*Note that we use asterisks to denote estimates and carets to denote unit vectors. Often, for example, when the subscript is $k|k$ or $k|k-1$, the asterisk will be left off when no confusion can occur.

$$\boldsymbol{\omega}(t) = \boldsymbol{\omega}^*(t) + \Delta\boldsymbol{\omega}(t) \quad , \tag{7}$$

and the predicted state vector, $\mathbf{x}^*(t)$, satisfies

$$\frac{d}{dt}\bar{q}^*(t) = \frac{1}{2}\,\Omega(\boldsymbol{\omega}^*(t))\,\bar{q}^*(t) \tag{8}$$

$$\frac{d}{dt}\boldsymbol{\omega}^*(t) = \mathbf{0} \tag{9}$$

The operation indicated in equation (6) is quaternion composition. Note that $\delta\mathbf{x}(t)$ is six-dimensional while $\mathbf{x}(t)$ is seven-dimensional.

The state-error vector satisfies

$$\frac{d}{dt}\Delta\boldsymbol{\theta}(t) = [[\boldsymbol{\omega}^*(t)]]\,\Delta\boldsymbol{\theta}(t) + \Delta\boldsymbol{\omega}(t) \quad , \tag{10}$$

$$\frac{d}{dt}\Delta\boldsymbol{\omega}(t) = \boldsymbol{\eta}(t) \quad , \tag{11}$$

where

$$[[\,\mathbf{v}\,]] \equiv \begin{bmatrix} 0 & v_3 & -v_2 \\ -v_3 & 0 & v_1 \\ v_2 & -v_1 & 0 \end{bmatrix} \quad . \tag{12}$$

On the basis of these equations, the state prediction equations become in standard Kalman filter notation

$$\bar{q}_{k|k-1} = M_{k-1}\,\bar{q}_{k-1|k-1} \quad , \tag{13}$$

$$\boldsymbol{\omega}_{k|k-1} = \boldsymbol{\omega}_{k-1|k-1} \quad , \tag{14}$$

where

$$M_{k-1} = \cos(\phi_{k-1}/2)\,I_{4\times4} + \sin(\phi_{k-1}/2)\,\Omega(\,\hat{\mathbf{n}}_{k-1}) \quad , \tag{15}$$

with the angle of rotation given by

$$\phi_{k-1} \equiv |\boldsymbol{\omega}^*_{k-1|k-1}|(t_k - t_{k-1}) \quad , \tag{16}$$

and the axis of rotation by

$$\hat{\mathbf{n}}_{k-1} \equiv \boldsymbol{\omega}^*_{k-1|k-1}/|\boldsymbol{\omega}^*_{k-1|k-1}| \quad . \tag{17}$$

Note that the estimated angular velocity is constant between filter updates as a result of the state equation and not a further approximation.

Covariance Prediction

The state error covariance matrix is defined as

$$P(t) = E\{\delta\mathbf{x}(t)\,\delta\mathbf{x}^T(t)\} \quad , \tag{18}$$

and the covariance prediction equation takes on the usual form

$$P_{k|k-1} = \Phi_{k-1}\,P_{k-1|k-1}\,\Phi^T_{k-1} + Q_{k-1} \quad . \tag{19}$$

To construct the state-error transition matrix we note that the state error vector satisfies

$$\frac{d}{dt}(\delta\mathbf{x}(t)) = F(t)\,\delta\mathbf{x}(t) + G(t)\boldsymbol{\eta}(t) \quad , \tag{20}$$

where the state-error sensitivity matrices are

$$F(t) = \begin{bmatrix} [[\boldsymbol{\omega}^*(t)]] & I_{3\times3} \\ 0_{3\times3} & 0_{3\times3} \end{bmatrix} \quad , \tag{21}$$

and

$$G(t) = \begin{bmatrix} 0_{3\times3} \\ I_{3\times3} \end{bmatrix} \quad . \tag{22}$$

Then the state-error transition matrix satisfies

$$\frac{d}{dt}\,\Phi(t,\, t') = F(t)\,\Phi(t,\, t') \quad , \tag{23}$$

with

$$\Phi(t',\, t') = I_{6\times6} \quad . \tag{24}$$

From Ref. 2 it follows that

$$\Phi_{k-1} \equiv \Phi(t_k,\, t_{k-1}) = \begin{bmatrix} \Theta_{k-1} & \Psi_{k-1} \\ 0_{3\times3} & I_{3\times3} \end{bmatrix} \quad , \tag{25}$$

with

$$\Theta_{k-1} = A(t_k)\,A^T(t_{k-1}) \quad , \tag{26}$$

$$\Psi_{k-1} = \int_{t_{k-1}}^{t_k} A(t_k)\,A^T(t')\,dt' \quad , \tag{27}$$

and $A(t)$ is the attitude matrix. Since $\boldsymbol{\omega}^*(t)$ is constant over the interval $(t_{k-1},\, t_k)$, it follows that

$$\Theta_{k-1} = I_{3\times3} + \sin(\phi_{k-1})\,[[\,\hat{\mathbf{n}}_{k-1}\,]] + (1 - \cos(\phi_{k-1}))\,[[\,\hat{\mathbf{n}}_{k-1}\,]]^2 \quad , \tag{28}$$

and

$$\Psi_{k-1} = a_{k-1}\,I_{3\times3} + b_{k-1}\,[[\,\hat{\mathbf{n}}_{k-1}\,]] + c_{k-1}\,[[\,\hat{\mathbf{n}}_{k-1}\,]]^2 \quad , \tag{29}$$

with

$$a_{k-1} = t_k - t_{k-1} \quad , \tag{30a}$$

$$b_{k-1} = \frac{1 - \cos(\phi_{k-1})}{\phi_{k-1}}\,(t_k - t_{k-1}) \quad , \tag{30b}$$

$$c_{k-1} = \left(1 - \frac{\sin(\phi_{k-1})}{\phi_{k-1}}\right)(t_k - t_{k-1}) \quad . \tag{30b}$$

The process noise covariance is given by

$$Q_{k-1} = \int_{t_{k-1}}^{t_k} \Phi(t_k,\, t')\,G(t')\,Q(t)\,G^T(t')\,\Phi^T(t_k,\, t')\,dt' \tag{31}$$

$$= \int_{t_{k-1}}^{t_k} \begin{bmatrix} L_{k-1}(t')\,Q(t')\,L_{k-1}^T(t') & L_{k-1}(t')\,Q(t') \\ Q(t')\,L_{k-1}^T(t') & Q(t') \end{bmatrix} dt' \quad , \tag{32}$$

where

$$L_{k-1} = a_{k-1}(t)\, I_{3\times3} + b_{k-1}(t)\,[[\,\hat{\mathbf{n}}_{k-1}\,]] + c_{k-1}(t)\,[[\,\hat{\mathbf{n}}_{k-1}\,]]^2 \quad , \tag{33}$$

with $a_{k-1}(t)$, $b_{k-1}(t)$, and $c_{k-1}(t)$ defined analogously to a_{k-1}, b_{k-1}, and c_{k-1} but with $t_k - t_{k-1}$ replaced by $t_k - t$. (Note that this change must be made in ϕ_{k-1} as well.)

THE MEASUREMENT MODEL AND THE FILTER UPDATE

The vector measurement model we consider is that of the QUEST model[3], which is

$$\hat{\mathbf{W}}_{i,k} = A_k\, \hat{\mathbf{V}}_{i,k} + \Delta\hat{\mathbf{W}}_{i,k} \quad , \tag{34}$$

where A_k is the attitude matrix at time t_k, $\hat{\mathbf{W}}_{i,k}$ is the observed star direction, $\hat{\mathbf{V}}_{i,k}$ is the reference star direction, and $\Delta\hat{\mathbf{W}}_{i,k}$ is the measurement noise, assumed to be Gaussian and white with covariance matrix, $R_{\hat{\mathbf{W}}_{i,k}}$, given by

$$R_{\hat{\mathbf{W}}_{i,k}} = \sigma^2_{\hat{\mathbf{W}}_{i,k}} \left[I - (A_k\hat{\mathbf{V}}_{i,k})(A_k\hat{\mathbf{V}}_{i,k})^T \right] \quad . \tag{35}$$

Here, k is the temporal index while i labels the sensor. The model of equations (34) and (35) has received frequent use.

An efficient algorithm, namely QUEST[2], exists for computing the maximum-likelihood estimate (MLE) of the attitude given the set of measurements $\{\hat{\mathbf{W}}_{i,k}, i = 1, \ldots, n_k\}$. QUEST furnishes an MLE quaternion, $\bar{p}_k^*$ and an estimate error covariance (defined in terms of $\Delta\boldsymbol{\theta}_k$, as above), which we may write as

$$\bar{p}_k^* = \delta\bar{p}_k \otimes \bar{q}_k \quad , \tag{36}$$

with

$$\delta\bar{p}_k = \frac{1}{\sqrt{1 + |\mathbf{v}_k|^2/4}} \begin{bmatrix} \mathbf{v}_k/2 \\ 1 \end{bmatrix} \quad . \tag{37}$$

Assuming that the individual star measurements are uncorrelated, $\mathbf{v}_k$ is a discrete white Gaussian process with covariance matrix R_k given by the QUEST algorithm as

$$R_k^{-1} = \sum_{i=1}^{n_k} \frac{1}{\sigma^2_{i,k}} \left[I - (A_k\hat{\mathbf{V}}_{i,k})(A_k\hat{\mathbf{V}}_{i,k})^T \right] \quad . \tag{38}$$

During the update step we define the state error $\delta\bar{q}_k$ according to

$$\bar{q}_k = \delta\bar{q}_k \otimes \bar{q}_{k|k-1} \quad . \tag{39}$$

If we define the derived measurement as

$$\bar{z}_k \equiv 2\,\bar{p}_k^* \otimes (\bar{q}_{k|k-1})^{-1} \quad , \tag{40}$$

then the vector components of this quantity satisfy simply

$$\mathbf{z}_k = \Delta\boldsymbol{\theta}_k + \mathbf{v}_k \quad , \tag{41}$$

$$= H_k\, \delta\mathbf{x}_k + \mathbf{v}_k \quad , \tag{42}$$

where

$$H_k = [I_{3\times3} \;\vdots\; 0_{3\times3}] \quad , \tag{43}$$

and

$$\delta\bar{q}_k \equiv \frac{1}{\sqrt{1+|\Delta\boldsymbol{\theta}_k|^2/4}} \begin{bmatrix} \Delta\boldsymbol{\theta}_k/2 \\ 1 \end{bmatrix} \quad , \tag{44}$$

which may now be implemented directly into the update step of the Kalman filter.

The Kalman filter update equations[4] are

$$B_k = H_k\, P_{k|k-1}\, H_k^T + R_k \tag{45}$$

$$= (P_{\theta\theta})_{k|k-1} + R_k \quad , \tag{46}$$

$$K_k = P_{k|k-1}\, H_k^T\, B_k^{-1} \quad , \tag{47}$$

$$\boldsymbol{\nu}_k = \mathbf{z}_k - H_k\, \delta\mathbf{x}_{k|k-1} \quad ,$$

$$= \mathbf{z}_k \quad , \tag{48}$$

$$\delta\mathbf{x}_{k|k} = \delta\mathbf{x}_{k|k-1} + K_k\, \boldsymbol{\nu}_k$$

$$= K_k\, \mathbf{z}_k \quad , \tag{49}$$

$$P_{k|k} = (I - K_k H_k)\, P_{k|k-1} \tag{50}$$

$$= (I - K_k H_k)\, P_{k|k-1}\, (I - K_k H_k)^T + K_k\, R_k\, K_k^T \quad . \tag{51}$$

Here, $\boldsymbol{\nu}_k$ is the innovation, B_k is the innovation covariance, and K_k is the Kalman gain matrix. The matrix $P_{\theta\theta}$ is the submatrix of the state-error covariance matrix between angle components. R_k is given by equation (38). Note that by definition the ***a priori*** state increment in equation (49) vanishes because it is measured from the predicted estimate. Once $\delta\mathbf{x}_{k|k}$ is computed $\bar{q}_{k|k}$ and $\boldsymbol{\omega}_{k|k}$ are computed using

$$\bar{q}_{k|k} = \delta\bar{q}_{k|k} \otimes \bar{q}_{k|k-1} \quad , \tag{52}$$

and

$$\boldsymbol{\omega}_{k|k} = \boldsymbol{\omega}_{k|k-1} + \Delta\boldsymbol{\omega}_{k|k} \quad . \tag{53}$$

INITIALIZATION

In general, no ***a priori*** initialization scheme is available for the filter since we do not know the statistical parameters of the distribution from which the initial condition is sampled. Thus, the initialization of the filter must be prior-free. Common practice in these cases has been to select an arbitrary initial condition with monstrously large initial covariance. This brutal methodology is dangerous and can result in poor convergence and large numerical errors if the system is poorly conditioned. Poor convergence properties would be particularly problematic in the Star Tracker mission, in which the attitude system will be reinitialized frequently. We wish to avoid the implementation of a square-root information filter, which would be a reliable, if more costly, alternative to the covariance filter presented here. Our approach instead is to compute the correct covariance given the filter model.

The QUEST algorithm provides complete attitudes at every time. Thus, it is sufficient to know these estimates at times t_1 and t_2 to estimate $\mathbf{x}_1$ and $\mathbf{x}_2$ unambiguously. Let us represent the two QUEST attitudes by the attitude matrices, C_1^* and C_2^*, rather than by the quaternion, $\bar{p}_1^*$ and $\bar{p}_2^*$. This will simplify the derivation because we are more accustomed to matrix multiplication than quaternion composition. Then,

$$C_1^* = e^{[\![\mathbf{v}_1]\!]}\, A_1 \quad , \tag{54}$$

$$C_2^* = e^{[\![\mathbf{v}_2]\!]}\, A_2 \quad , \tag{55}$$

where the $\mathbf{v}_k$ are the same as defined earlier. These measurements have realizations $C_1^{*\prime}$ and $C_2^{*\prime}$, the specific values given by QUEST*. The dynamical equations provide *uncertain* constraints on A_1, $A_2, \boldsymbol{\omega}_1$, and $\boldsymbol{\omega}_2$, namely

$$A_2 = e^{[[\Delta\boldsymbol{\theta}_1]]}\, e^{[[\boldsymbol{\omega}_1 (t_2 - t_1)]]}\, A_1 \quad , \tag{56}$$

$$\boldsymbol{\omega}_2 = \boldsymbol{\omega}_1 + \Delta\boldsymbol{\omega}_1 \quad , \tag{57}$$

and

$$\begin{bmatrix} \Delta\boldsymbol{\theta}_1 \\ \Delta\boldsymbol{\omega}_1 \end{bmatrix} \sim \mathcal{N}(\mathbf{0}, \mathcal{Q}_1) \quad . \tag{58}$$

(Note that the calculation of $\mathcal{Q}_1$ requires that we know $\boldsymbol{\omega}_{1|1}$, which must itself come from the initialization. This turns out not to be a problem since the calculation of $\boldsymbol{\omega}_{1|1}$ (actually of the smoothed estimate $\boldsymbol{\omega}_{1|2}$) in the initialization step can be carried out before that of $\mathcal{Q}_1$.)

If we solve equations (56) and (57) for A_1 and $\boldsymbol{\omega}_1$ and substitute these in equations (54) and (55), we will obtain two very non-linear stochastic equations for A_2 and $\boldsymbol{\omega}_2$, namely,

$$C_1^* = e^{[[\mathbf{v}_1]]}\, e^{-[[\boldsymbol{\omega}_2 - \Delta\boldsymbol{\omega}_1]]\Delta t}\, e^{-[[\Delta\boldsymbol{\theta}_1]]}\, A_2 \quad , \tag{59}$$

$$C_2^* = e^{[[\mathbf{v}_2]]}\, A_2 \quad . \tag{60}$$

Where we have written Δt for $t_2 - t_1$. Substituting equation (60) into equation (59) yields

$$C_1^*\, C_2^{*T} = e^{[[\mathbf{v}_1]]}\, e^{-[[\boldsymbol{\omega}_2 - \Delta\boldsymbol{\omega}_1]]\Delta t}\, e^{-[[\Delta\boldsymbol{\theta}_1]]}\, e^{-[[\mathbf{v}_2]]} \quad , \tag{61}$$

which depends only on $\boldsymbol{\omega}_2$, while equation (60) depends only on A_2. We now define $\boldsymbol{\beta}$, which has units of angular velocity, by

$$C_2^{*\prime}\, C_1^{*\prime T} \equiv e^{[[\boldsymbol{\beta}]]\Delta t} \quad , \tag{62}$$

If we reparameterize the state now in terms of $\Delta\boldsymbol{\varphi}_2$ and $\Delta\boldsymbol{\tau}_2$ defined by

$$A_2 = e^{[[\Delta\boldsymbol{\varphi}_2]]}\, C_2^{*\prime} \quad , \tag{63}$$

$$\boldsymbol{\omega}_2 = \boldsymbol{\beta} + \Delta\boldsymbol{\tau}_2 \quad , \tag{64}$$

then we obtain in terms of new measurement matrices, Z_1 and Z_2 ,

$$\begin{aligned} Z_1 &\equiv C_2^*\, C_2^{*\prime T} \\ &= e^{[[\mathbf{v}_2]]}\, e^{[[\Delta\boldsymbol{\varphi}_2]]} \quad , \end{aligned} \tag{65}$$

$$\begin{aligned} Z_2 &\equiv C_2^{*\prime}\, C_1^{*\prime T}\, C_1^*\, C_2^{*T} \\ &= e^{[[\boldsymbol{\beta}]]\Delta t}\, e^{[[\mathbf{v}_1]]}\, e^{-[[\boldsymbol{\beta} + \Delta\boldsymbol{\tau}_2 - \Delta\boldsymbol{\omega}_1]]\Delta t}\, e^{-[[\Delta\boldsymbol{\theta}_1]]}\, e^{-[[\mathbf{v}_2]]} \quad . \end{aligned} \tag{66}$$

Note that we have defined Z_1 and Z_2 so that each has realization (Z_1' and Z_2') $I_{3\times3}$.

We wish to linearize these two effective measurements by combining the exponents, taking the matrix logarithms, and extracting the arguments of the antisymmetric matrices. This operation cannot take place yet because $\boldsymbol{\beta}$ is not infinitesimal. However, it is true that

$$e^{[[\boldsymbol{\beta}]]\Delta t}\, e^{[[\mathbf{v}_1]]}\, e^{-[[\boldsymbol{\beta}]]\Delta t} = e^{[[T_{\boldsymbol{\beta}}\mathbf{v}_1]]} \quad , \tag{67}$$

where

$$T_{\boldsymbol{\beta}} = e^{[[\boldsymbol{\beta}]]\Delta t} \quad , \tag{68}$$

*In the remainder of this report, primed quantities will indicate realizations and unprimed quantities the random variables.

which follows immediately from the transformation properties of orthogonal matrices, and

$$e^{[[\boldsymbol{\beta}+\Delta\boldsymbol{\tau}_2-\Delta\boldsymbol{\omega}_1]]\Delta t} = e^{[[\Delta\boldsymbol{\beta}]]\Delta t}\, e^{[[\boldsymbol{\beta}]]\Delta t} \quad , \tag{69}$$

where

$$\Delta\boldsymbol{\beta} = D^{-1}(\boldsymbol{\beta}\Delta t)\,(\Delta\boldsymbol{\tau}_2 - \Delta\omega_1) \quad , \tag{70}$$

$$D(\boldsymbol{\beta}\Delta t) = I - \frac{1}{2}[[\boldsymbol{\beta}\Delta t]] + \frac{2-\beta\Delta t\,\cot((\beta\Delta t)/2)}{2\,(\beta\Delta t)^2}[[\boldsymbol{\beta}\Delta t]]^2 \quad , \tag{71}$$

and β is the magnitude of $\boldsymbol{\beta}$. (This formula is derived in the Appendix.) Using these expressions in equation (66), all of the exponents become infinitesimal with the equivalent vectorial linearized result

$$\boldsymbol{\zeta}_1 = \Delta\boldsymbol{\varphi}_2 + \mathbf{v}_2 \quad , \tag{72}$$

$$\boldsymbol{\zeta}_2 = -D^{-1}(\boldsymbol{\beta}\Delta t)\,(\Delta\boldsymbol{\tau}_2 - \Delta\boldsymbol{\omega}_1)\Delta t + T_\beta\,\mathbf{v}_1 - \mathbf{v}_2 - \Delta\boldsymbol{\theta}_1 \quad . \tag{73}$$

Since $\boldsymbol{\zeta}_1$ and $\boldsymbol{\zeta}_2$ have realization $\mathbf{0}$, it follows immediately that the maximum likelihood estimates of $\Delta\boldsymbol{\varphi}_2$ and $\Delta\boldsymbol{\tau}_2$ vanish. Hence, the initialization of the state vector in the Kalman filter at t_2 is

$$A_{2|2} = C_2^* \quad , \tag{74}$$

$$\boldsymbol{\omega}_{2|2} = \boldsymbol{\beta} \quad , \tag{75}$$

The covariance of these estimates is $P_{2|2}$, which is readily calculated as the covariance of

$$\delta\tilde{\mathbf{x}}_{2|2} \equiv \delta\mathbf{x}_2 - \delta\mathbf{x}_{2|2} = \begin{bmatrix} \mathbf{v}_2 \\ -\Delta\boldsymbol{\omega}_1 - (\Delta t)^{-1}\,D(\boldsymbol{\beta}\Delta t)\,(T_\beta\,\mathbf{v}_1 - \mathbf{v}_2 - \Delta\boldsymbol{\theta}_1) \end{bmatrix} \tag{76}$$

using equations (38) and (58). Smoothed values of the state variables, $A_{1|2}$ and $\boldsymbol{\omega}_{1|2}$, can be obtained by predicting the estimates at t_2 backwards in time. However, it is easily seen in the present case that

$$A_{1|2} = C_1^* \quad , \tag{77}$$

$$\boldsymbol{\omega}_{1|2} = \boldsymbol{\omega}_{2|2} = \boldsymbol{\beta} \quad . \tag{78}$$

Strickly speaking, the computation of Q_1 requires that we know $\boldsymbol{\omega}_{1|1}$. However, the estimate $\boldsymbol{\omega}_{2|2}$, will be a sufficiently close approximation for calculating Q_1.

As an approximation we note that neglecting the terms in the initial covariance in Q_1, which should be smaller than those in R_1 and R_2 if any benefit is to be derived from the Kalman filter, and the change in the attitude from t_1 to t_2, the updated covariance at t_2 becomes approximately

$$P_{2|2} \simeq \begin{bmatrix} R_2 & (1/\Delta t)\,R_2 \\ (1/\Delta t)\,R_2 & (1/(\Delta t)^2)\,(R_1 + R_2) \end{bmatrix} \quad , \tag{79}$$

whose implementation in much simpler.

SIMULATION RESULTS

The filter has been tested using a dynamical model in which the attitude is given by the Gibbs vector[5]

$$\mathbf{g}(t) = \mathbf{g}_2(t) \circ \mathbf{g}_1(t) \quad , \tag{80}$$

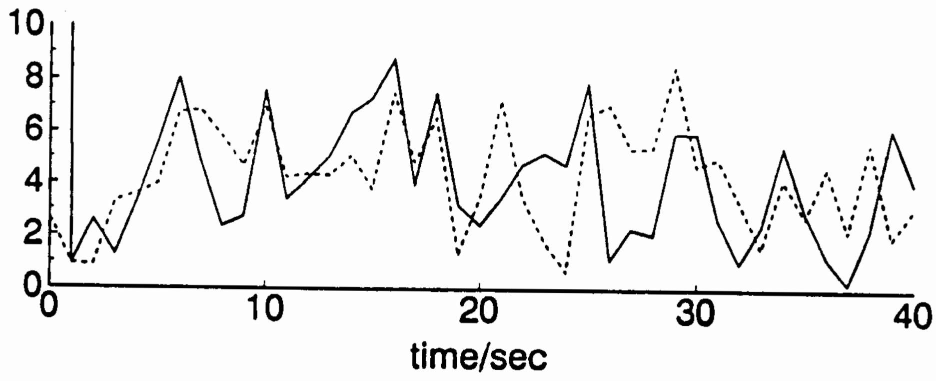

Fig. 1 Star Tracker Pointing Errors

The solid line gives the filter result. The dashed line is the single-frame estimate using QUEST. The angles are given in arc sec.

where "∘" denotes Gibbs-vector composition and

$$\mathbf{g}_1(t) = \hat{\mathbf{z}}\ \tan(\omega_o t/2) \quad , \tag{81}$$

with $\omega_o = .001\,\mathrm{sec}^{-1}$ is just one rotation per orbit motion of the spacecraft and

$$\mathbf{g}_2(t) = \hat{\mathbf{x}}\ \tan(a_1 \sin\lambda_1 t) + \hat{\mathbf{y}}\ \tan(a_2 \cos\lambda_2 t) + \hat{\mathbf{z}}\ \tan(a_3 \sin\lambda_3 t) \quad , \tag{82}$$

with

$$a_1 = 2.\mathrm{deg} \quad , \quad a_2 = 1.\mathrm{deg} \quad , \quad a_3 = 3.\mathrm{deg} \quad , \tag{83a}$$

$$\lambda_1 = .02\,\mathrm{sec}^{-1} \quad , \quad \lambda_2 = .025\,\mathrm{sec}^{-1} \quad ,\mathrm{and} \quad \lambda_3 = .015\,\mathrm{sec}^{-1} \quad , \tag{83b}$$

an arbitrary high-frequency component of the attitude motion which is roughly at the limits of angular velocity and acceleration expected for the normal operation of the Star Tracker mission. We assume conservatively that the star tracker has an accuracy of 10 arc sec per axis and that the instrument is sampled at one-second intervals for 40 seconds. The model power spectral density was chosen to be a multiple of the identity matrix with the constant of proportionality adjusted so that

$$\mathrm{trace}\, Q\Delta t = \mathrm{trace}\left\langle \delta\boldsymbol{\omega}(t)\delta\boldsymbol{\omega}^T(t)\right\rangle \quad , \tag{84}$$

where $< \cdot >$ denotes a time average and

$$\delta\boldsymbol{\omega}(t) \equiv \boldsymbol{\omega}(t+\Delta t) - \boldsymbol{\omega}(t) \simeq 2\frac{d^2\mathbf{g}_2(t)}{dt^2}\Delta t \quad . \tag{85}$$

Figures 1 and 2 show plots of the attitude estimate errors as a function of time. The pointing error is defined as the angle (defined to be positive) between the true and the estimated bore-sight directions. The roll error is the magnitude of the error in the angle of rotation about the bore sight. In these curves the solid line gives the filter result and the dashed line gives the single-frame result from QUEST. Figure 3 shows the estimate errors for the three components of the angular velocity. There is a large improvement in the roll error using the filter since the process noise covariance is small compared with that of the single frame roll errors. The improvement in the pointing error is less noticeable because the process noise covariance is large compared with that of the pointing error.

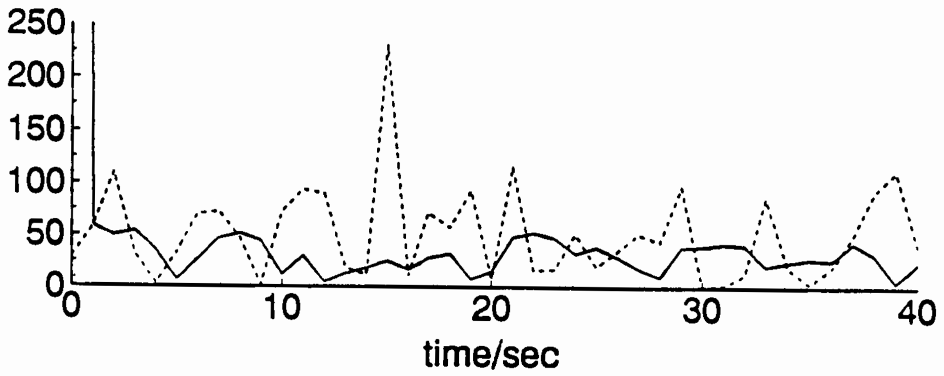

Fig. 2 Star Tracker Roll Errors

The solid line gives the filter result. The dashed line is the single-frame estimate using QUEST. The angles are given in arc sec.

We have also compared the three initialization methods. These are: (i) the statistically correct initialization scheme of Equations (74)–(76); (ii) the statistically and dynamically approximate scheme of Equations (74), (75), and (79); and (iii) the more common approximate scheme in which $P_{1|0}$ is set to

$$P_{1|0} = \begin{bmatrix} \sigma_{\theta 0}^2 \, I_{3\times 3} & 0_{3\times 3} \\ 0_{3\times 3} & \sigma_{\omega 0}^2 \, I_{3\times 3} \end{bmatrix} \quad , \tag{86}$$

where $\sigma_{\theta 0}$ and $\sigma_{\omega 0}$ are large numbers. Here we have chosen $\sigma_{\theta 0} \simeq 1000.$ deg, and $\sigma_{\omega 0} \simeq 1000.$ deg/sec. Beyond the first frame of data the three methods turn out to be indistinguishable.

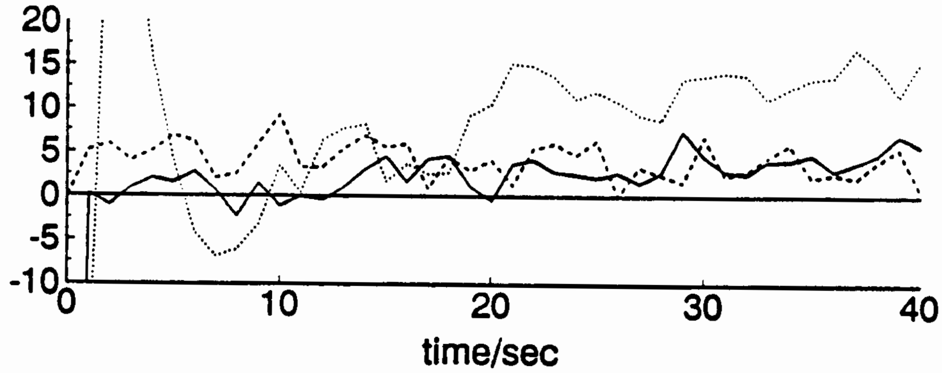

Fig. 3 Angular Velocity Estimate Errors

The solid line gives the x-component, the dashed line the y-component, and the dotted line the z-component. The rates are given in arc sec/sec.

DISCUSSION AND CONCLUSIONS

An efficient Kalman filter has been developed for a spacecraft for which attitude information is supplied solely by a true star camera, i.e., capable of sighting many stars simultaneously. The Kalman filter uses the QUEST algorithm to compute a sufficient statistic which provides a means of data compression within the filter. In addition, an algorithm is developed for initializing the filter at the correct prior-free estimate and covariance, rather than relying on brute-force initialization strategies and hoping that the filter will converge quickly enough. In the present example, however, it turns out that the brute-force initialization strategy works very well. A great advantage of the present method is that the sufficient statistic, being an attitude, can be compared with the filtered attitude as a check on consistency. This methodology, then, provides a safety net for mission analysts who have learned to be wary of Kalman filters in attitude applications.

The lack of an accurate dynamic model or attitude-rate sensors poses important restrictions on the utility of the filter. If the uncertainty in the attitude due to the unmodeled dynamics is greater than $(Q\Delta t)^{1/2}$, and in this case the unmodeled dynamics is the entire dynamics of the system, then the Kalman filter will simply provide an attitude estimate which is close in accuracy to the single frame result. Thus, the filter proposed here will have limited usefulness in situations where the spacecraft is very agile. It is expected for the Star Tracker mission, however, that this limitation will not be significant during normal operation.

The idea of using a least-squares estimator as a preprocessor for a Kalman filter is not new and originated in earlier research on autonomous attitude determination systems[6], which, in fact, considered also the use of CCD star trackers. An implementation for a gravity-gradient stabilized spacecraft of a two-step filter with a QUEST preprocessor and a dynamical model for attitude prediction has been considered by Varotto[7], whose treatment is similar in spirit to the present work but very different in execution. It may be noted that the use of sufficient statistics as data compressors for Kalman filters, of which the present work is an example, is not new. Extensive references may be found in Ref. 8.

APPENDIX – ROTATION VECTOR ALGEBRA

The rotation vector is defined as

$$\boldsymbol{\theta} = \theta\,\hat{\mathbf{n}} \quad , \tag{A1}$$

where θ is the angle of rotation and $\hat{\mathbf{n}}$ is the axis of rotation. The rotation vector is related to the attitude matrix by

$$A(\boldsymbol{\theta}) = \exp\{[[\boldsymbol{\theta}]]\} \quad , \tag{A2}$$

where exp denotes matrix exponentiation and is defined by the Taylor series for the exponential function. Evaluating the Taylor series leads to the usual expression

$$A(\boldsymbol{\theta}) = I_{3\times3} + \frac{\sin\theta}{\theta}[[\boldsymbol{\theta}]] + \frac{1-\cos\theta}{\theta^2}[[\boldsymbol{\theta}]]^2 \quad , \tag{A3}$$

where $\theta = |\boldsymbol{\theta}|$. The inverse relation is most easily obtained by solving for θ using

$$\mathrm{trace}\,A = 1 + 2\cos\theta \quad , \tag{A4}$$

and

$$[[\boldsymbol{\theta}]] = \left(\frac{\theta}{2\sin\theta}\right)(A - A^T) \quad . \tag{A5}$$

The kinematic equation for the rotation vector is

$$\frac{d}{dt}\boldsymbol{\theta}(t) = D(\boldsymbol{\theta})\boldsymbol{\omega}(t) \quad , \tag{A6}$$

where

$$D(\boldsymbol{\theta}) \equiv I_{3\times3} - \frac{1}{2}[[\boldsymbol{\theta}]] + \frac{2 - \theta\cot(\theta/2)}{2\theta^2}[[\boldsymbol{\theta}]]^2 \quad , \tag{A7}$$

Thus, if $\Delta\boldsymbol{\theta}$ is the rotation vector for an infinitesimal rotation, we may define an addition operation for the rotation vector when the second vector is infinitesimal, which is

$$\boldsymbol{\theta} \oplus \Delta\boldsymbol{\theta} \equiv \boldsymbol{\theta} + D(\boldsymbol{\theta})\Delta\boldsymbol{\theta} \quad . \tag{A8}$$

The operation $\oplus$ is not commutative. In a similar fashion we may define the difference of two rotation vectors $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ when the difference is infinitesimal by insisting that

$$\boldsymbol{\theta}_2 = \boldsymbol{\theta}_1 \oplus (\boldsymbol{\theta}_2 \ominus \boldsymbol{\theta}_1) \quad , \tag{A9}$$

which leads to

$$\boldsymbol{\theta}_2 \ominus \boldsymbol{\theta}_1 = D^{-1}(\boldsymbol{\theta}_1)(\boldsymbol{\theta}_2 - \boldsymbol{\theta}_1) \quad . \tag{A10}$$

As an example we may consider

$$\boldsymbol{\theta}_1 = \boldsymbol{\beta}\Delta t \quad , \quad \boldsymbol{\theta}_2 = (\boldsymbol{\beta} + \Delta\boldsymbol{r}_2 - \Delta\boldsymbol{\omega}_1)\Delta t \quad , \tag{A11}$$

which leads to equation (70) above.

ACKNOWLEDGEMENTS

This project is supported under contract by the U. S. Army Engineer Topographic Laboratories, Fort Belvoir, Virginia. The encouragement and support of Connie Gray is gratefully acknowledged.

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