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EULER-2 AND *EULER-n* ALGORITHMS FOR ATTITUDE DETERMINATION FROM VECTOR OBSERVATIONS*

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Abstract—This paper presents two new mathematical approaches to the optimal spacecraft attitude determination based on vector observations: *EULER*-2 and *EULER-n*. They compute the optimal Euler axis and angle by a deterministic and an iterative method, respectively. *EULER*-2, whose application is restricted when only two unit-vector pairs are available ($n = 2$), is based on the attitude matrix rotation property and on a demonstrated co-planarity condition. This co-planarity condition allows also the extension of the existing *TRIAD* algorithm to the optimal solution, getting *TRIAD*-2. The *EULER-n* algorithm, which has no limitation on the unit-vector number n, uses a recursive equation which converges with only one or two iterations when using *EULER*-2 for evaluating the starting point. Some numerical tests on accuracy and computational speed are performed for *EULER*-2, *TRIAD*-2, and *EULER-n*. Plots show that all these three new optimal attitude estimation algorithms are particularly suitable for fast optimal attitude determination. © 1997 Elsevier Science Ltd.

INTRODUCTION

The rotation representation by means of the Euler's axis and angle provides, as compared to other methods (quaternion, Gibb's vector, Euler's angles, etc.), the undoubted advantage of supplying in an explicit way, the physical parameters of the rotation itself. The attitude matrix A, expressed by the rotation axis (Euler axis e) and angle (Euler angle Φ)

$$A = I \cos \Phi + (1 - \cos \Phi) e e^{\mathrm{T}} - \tilde{e} \sin \Phi \qquad (1)$$

can be considered the rotation operator which (in the ideal case), rotates the reference unit-vector v to the observed unit-vector s, that is $Av = s$ (Fig. 1).

The optimal attitude determination problem consists of the estimation of the attitude representation (matrix, quaternion, Euler axis and angle, Euler angles, Gibb's vector, etc.) by respecting a "best criteria", as those introduced by Wahba (1965) [1], and by using a data set consisting of $n \geq 2$ unit-vector pair $(s_i\ v_i)$, indicating the same ith direction in two different reference systems, and with the precision of each pair given by relative weights α_i, such that $\Sigma_i \alpha_i = 1$.

EULER-2: THE IDEAL CASE

Let us consider when $n = 2$ and the ideal case

$$A v_1 = s_1,\ A v_2 = s_2\,, \qquad (2)$$

that is when $s_1^{\mathrm{T}} s_2 = v_1^{\mathrm{T}} v_2$ is verified. In this ideal case the previous equation states that the measurement unit-vectors (s_1, s_2) are obtained by the rotation of the reference unit-vectors (v_1, v_2) about e by the angle Φ. This means v_1 and v_2 each lie on a cone about e. Therefore, we can write

$$\begin{cases} e^{\mathrm{T}} v_1 = e^{\mathrm{T}} s_1 \\ e^{\mathrm{T}} v_2 = e^{\mathrm{T}} s_2 \end{cases} \Rightarrow \begin{cases} e^{\mathrm{T}}(v_1 - s_1) = 0 \\ e^{\mathrm{T}}(v_2 - s_2) = 0 \end{cases} \qquad (3)$$

This means that the Euler axis e is perpendicular to both vectors $(v_1 - s_1)$ and $(v_2 - s_2)$. Therefore the Euler axis and angle (e, Φ) can be evaluated by using

$$\begin{cases} e = \pm \dfrac{(v_1 - s_1) \times (v_2 - s_2)}{|(v_1 - s_1) \times (v_2 - s_2)|} \end{cases}$$

$$\begin{cases} \sin \Phi = z^{\mathrm{T}} e w \\ \cos \Phi = z^{\mathrm{T}} w, \end{cases} \qquad (4)$$

where $w = s_i - (s_i^{\mathrm{T}} e) e$, $z = v_i - (v_i^{\mathrm{T}} e) e$ and the subscript i can be either 1 or 2. The attitude matrix A is then obtained by using equation (1).

EULER-2: THE REAL CASE

In general, that is when considering the real case, the condition $s_1^{\mathrm{T}} s_2 = v_1^{\mathrm{T}} v_2$ is not verified. Therefore, it is not possible to devise such an attitude matrix which satisfies both conditions (2). In this case, however, it is possible to compute a matrix A^* which optimally estimates the attitude matrix A by satisfying Wahba's "optimal condition"

$$\sigma_w = \sum_i \alpha_i s_i^{\mathrm{T}} A^* v_i = \sum_i \alpha_i s_i^{\mathrm{T}} x_i = \sum_i \alpha_i \cos \vartheta_i = \max \qquad (5)$$

where we have set $A^* v_i = x_i \neq s_i$, and therefore $x_1^{\mathrm{T}} x_2 = v_1^{\mathrm{T}} v_2$. Hereafter it will be demonstrated that, when $n = 2$, the plane defined by the unit-vectors x_1 and x_2 is the same as that of s_1 and s_2. This implies a co-planarity condition between s_1, s_2, x_1 and x_2.

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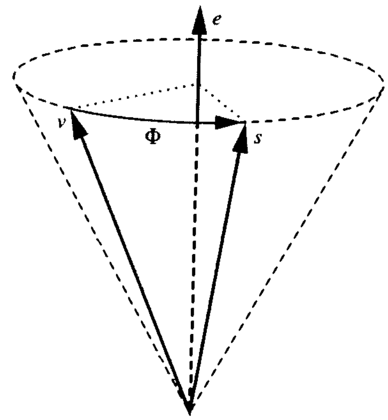

Fig. 1. Rotational property of attitude matrix.

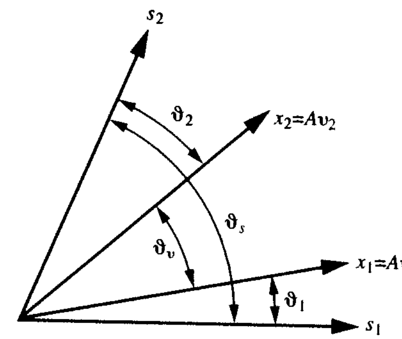

Fig. 3. Co-plenarity condition.

In order to demonstrate this, let us for the moment consider the unit-vector x_2 as fixed in space.

Therefore, referring to the Fig. 2, let us compute x_1 which maximizes the Wahba Loss function (5). The problem is to maximize σ_w with the two constraints of a fixed angle between x_1 and x_2 ($x_2^T x_1 = v_2^T v_1 = \cos \vartheta_v$), and x_1 be a unit-vector ($x_1^T x_1 = 1$). By using the Lagrangian multiplier technique the augmented cost function is

$$\sigma_w^* = \alpha_1 s_1^T x_1 + \alpha_2 s_2^T x_2 - \lambda_a(x_2^T x_1 - \cos \vartheta_v) - \lambda_b(x_1^T x_1 - 1) = \max\,. \quad (6)$$

By taking the derivative we obtain the condition

$$\frac{d\sigma_w^*}{dx_1} = \alpha_1 s_1 - \lambda_a x_2 - 2\lambda_b x_1 = 0\,, \quad (7)$$

which implies the solution

$$x_1 = (\alpha_1 s_1 - \lambda_a x_2)/(2\lambda_b)\,. \quad (8)$$

This means x_1 can be linearly expressed by the unit-vectors s_1 and x_2, and consequently x_1 is co-planar to them. Therefore, wherever the unit-vector x_2 is pointing (right or wrong direction), the optimal condition is obtained when s_1, x_1 and x_2 are co-planar. In the same way it is possible to demonstrate that wherever the unit-vector x_1 is pointing the unit-vectors s_2, x_1 and x_2 must also be co-planar.

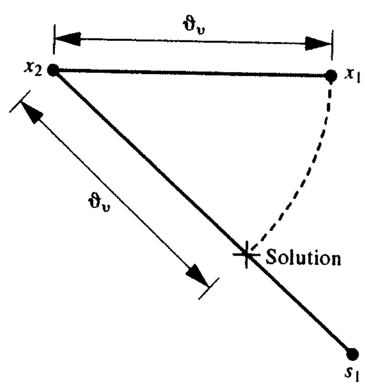

Fig. 2. Solution for x_1 unit-vector.

Therefore, in order to satisfy both conditions the unit-vectors x_1 and x_2 must to lie on the plane of s_1 and s_2 (Fig. 3). Based on this co-planarity condition we can now evaluate the x_1 and x_2 directions.

The optimization, in the Wahba sense, implies

$$\sigma_w = \alpha_1 \cos \vartheta_1 + \alpha_2 \cos \vartheta_2 = \max \quad (9)$$

Since $\vartheta_2 = \vartheta_s - \vartheta_v - \vartheta_1$ (Fig. 3), this condition is satisfied when

$$\alpha_1 \sin \vartheta_1 = \alpha_2 \sin \vartheta_2 \quad (10)$$

equations (9) and (10) lead to

$$\begin{cases} \sin \vartheta_1 = \alpha_2 \sin(\vartheta_s - \vartheta_v) \\ \cos \vartheta_1 = \alpha_1 + \alpha_2 \cos(\vartheta_s - \vartheta_v) \end{cases} \quad (11)$$

When ϑ_1 has been computed (and therefore also $\vartheta_2 = \vartheta_s - \vartheta_v - \vartheta_1$) it is possible to evaluate

$$\begin{cases} x_1 = \dfrac{s_1 \sin(\vartheta_v + \vartheta_2) + s_2 \sin \vartheta_1}{\sin \vartheta_s} \\ x_2 = \dfrac{s_2 \sin(\vartheta_v + \vartheta_1) + s_1 \sin \vartheta_2}{\sin \vartheta_s} \end{cases} \quad (12)$$

Once x_1 and x_2 have been evaluated, the optimal estimation of the Euler axis and angle is given by

$$\begin{cases} e^* = \pm \dfrac{(v_1 - x_1) \times (v_2 - x_2)}{|(v_1 - x_1) \times (v_2 - x_2)|} \end{cases}$$

$$\begin{cases} \sin \Phi^* = z^T \tilde{e}^* w \\ \cos \Phi^* = z^T w, \end{cases} \quad (13)$$

where $w = x_i - (x_i^T e^*)e^*$, $z = v_i - (v_i^T e^*)e^*$ and the subscript i can be either 1 or 2. The optimal attitude matrix estimation A^* is then obtained by using equation (1).

An alternative way to compute x_1 and x_2 can be devised by first computing $n_s = s_1 \times s_2/|s_1 \times s_2|$. Then, the unit-vector x_2 can also be expressed as $x_2 = R(n_s, \vartheta_v)x_1$ where

$$R(n_s, \vartheta_v) = I \cos \vartheta_v + (1 - \cos \vartheta_v) n_s n_s^T + \tilde{n}_s \sin \vartheta_v \quad (14)$$

is the rotation matrix about $\boldsymbol{n}_s$ of an angle ϑ_r. Wahba's condition may also be written as

$$\sigma_w = \alpha_1 \boldsymbol{s}_1^T \boldsymbol{x}_1 + \alpha_2 \boldsymbol{s}_2^T \boldsymbol{R}(\boldsymbol{n}_s, \vartheta_r) \boldsymbol{x}_1 = \boldsymbol{y}^T \boldsymbol{x}_1 = \max \quad (15)$$

In order to maximize σ_w the unit-vector $\boldsymbol{x}_1$ must therefore be parallel to $\boldsymbol{y}$. This means $\boldsymbol{x}_1 = \boldsymbol{y}/|\boldsymbol{y}|$ and then $\boldsymbol{x}_2 = \boldsymbol{R}(\boldsymbol{n}_s, \vartheta_r)\boldsymbol{x}_1$.

TRIAD-2: TRIAD OPTIMIZATION

The main idea of the *TRIAD* attitude determination algorithm [2] is based on devising two triads $\boldsymbol{V} = [\boldsymbol{v}_x\ \boldsymbol{v}_y\ \boldsymbol{v}_z]$ and $\boldsymbol{S} = [\boldsymbol{s}_x\ \boldsymbol{s}_y\ \boldsymbol{s}_z]$ constructed using the reference and observed vectors, respectively. The matrices $\boldsymbol{V}$ and $\boldsymbol{S}$ can be devised in any fashion, provided that $\boldsymbol{A}^*\boldsymbol{V} = \boldsymbol{S}$ and the matrix $\boldsymbol{V}$ is not singular (this means that the three vectors $\boldsymbol{v}_x$, $\boldsymbol{v}_y$ and $\boldsymbol{v}_z$ are not co-planar, that is $\boldsymbol{v}_x^T \tilde{\boldsymbol{v}}_y \boldsymbol{v}_z \neq 0$). If $\boldsymbol{V}$ and $\boldsymbol{S}$ are chosen be orthonormals, as in Ref. [2], the solution is given by $\boldsymbol{A}^* = \boldsymbol{S}\boldsymbol{V}^T$.

Based on the co-planarity condition between $\boldsymbol{s}_1$, $\boldsymbol{x}_1$, $\boldsymbol{s}_2$ and $\boldsymbol{x}_2$, it is possible to enhance the *TRIAD* algorithm to the optimal solution. As a result of this property the directions indicated by $\boldsymbol{s}_1 \times \boldsymbol{s}_2$ and $\boldsymbol{A}^*(\boldsymbol{v}_1 \times \boldsymbol{v}_2)$ are the same. One therefore selects these directions to belong to the two triads. Therefore, let us choose

$$\boldsymbol{s}_y = \boldsymbol{n}_s = \frac{\boldsymbol{s}_1 \times \boldsymbol{s}_2}{|\boldsymbol{s}_1 \times \boldsymbol{s}_2|}, \ \boldsymbol{v}_y = \boldsymbol{n}_v = \frac{\boldsymbol{v}_1 \times \boldsymbol{v}_2}{|\boldsymbol{v}_1 \times \boldsymbol{v}_2|}. \quad (16)$$

As already noted, the unit-vectors $\boldsymbol{x}_1 = \boldsymbol{A}^*\boldsymbol{v}_1$ and $\boldsymbol{x}_2 = \boldsymbol{A}^*\boldsymbol{v}_2$ do not coincide with $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$, but they make angles ϑ_1 and ϑ_2, respectively (Fig. 3). This means that, if we had chosen $\boldsymbol{v}_1$ as one of the reference triad components, the corresponding observed triad component should not have been $\boldsymbol{s}_1$, but $\boldsymbol{x}_1$ instead. The latter can be regarded as the $\boldsymbol{s}_1$ vector rotated by the angle ϑ_1 about the $\boldsymbol{n}_s$ direction. In other words $\boldsymbol{x}_1 = \boldsymbol{R}(\boldsymbol{n}_s, \vartheta_1)\boldsymbol{s}_1$. Let us then choose $\boldsymbol{v}_x = \boldsymbol{v}_1$ and $\boldsymbol{s}_x = \boldsymbol{x}_1 = \boldsymbol{R}(\boldsymbol{n}_s, \vartheta_1)\boldsymbol{s}_1$. The third axis is thus defined as the one forming an orthonormal matrix.

Therefore the optimal solution is given by

$$\begin{aligned} \boldsymbol{A}^* &= [\boldsymbol{x}_1\ \boldsymbol{n}_s\ \boldsymbol{x}_1 \times \boldsymbol{n}_s][\boldsymbol{v}_1\ \boldsymbol{n}_v\ \boldsymbol{v}_1 \times \boldsymbol{n}_v]^T \\ &= \boldsymbol{R}(\boldsymbol{n}_s, \vartheta_1)[\boldsymbol{s}_1\ \boldsymbol{n}_s\ \boldsymbol{s}_1 \times \boldsymbol{n}_s][\boldsymbol{v}_1\ \boldsymbol{n}_v\ \boldsymbol{v}_1 \times \boldsymbol{n}_v]^T \quad (17) \end{aligned}$$

which extends the original *TRIAD* formulation to the optimal attitude matrix computation.

Numerical tests on speed computation demonstrate that the choice of having orthonormal triads (in order of not loading the computation by adding the inversion of the $\boldsymbol{V}$-matrix), is not the best one. Therefore, hereafter are listed two other possible solution forms (the first of which is the fastest one and therefore that selected for speed tests) which do not use orthonormal triads

$$\boldsymbol{A}^* = \begin{cases} = [\boldsymbol{x}_1\ \boldsymbol{x}_2\ \boldsymbol{x}_1 \times \boldsymbol{x}_2][\boldsymbol{v}_1\ \boldsymbol{v}_2\ \boldsymbol{v}_1 \times \boldsymbol{v}_2]^{-1} \\ = [\boldsymbol{e}\ \boldsymbol{n}_s\ \boldsymbol{e} \times \boldsymbol{n}_s][\boldsymbol{e}\ \boldsymbol{n}_v\ \boldsymbol{e} \times \boldsymbol{n}_v]^{-1} \end{cases} \quad (18)$$

EULER-N

The general expression (for n unit-vectors pairs) of the Wahba's cost function expressed in term of Euler axis and angle is

$$\begin{aligned} \sigma_w &= \cos\Phi\ \mathrm{tr}[\boldsymbol{B}] \\ &\quad + (1 - \cos\Phi)\boldsymbol{e}^T\boldsymbol{B}\boldsymbol{e} + \sin\Phi\ \boldsymbol{f}^T\boldsymbol{e} = \max \quad (19) \end{aligned}$$

where

$$\begin{cases} \boldsymbol{B} = \sum_i \alpha_i \boldsymbol{s}_i \boldsymbol{v}_i^T = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \\ \boldsymbol{f} = \begin{Bmatrix} b_{23} - b_{32} \\ b_{31} - b_{13} \\ b_{12} - b_{21} \end{Bmatrix} \end{cases} \quad (20)$$

The Euler axis constraint leads to the augmented cost function

$$\sigma_w^* = \sigma_w - \lambda(\boldsymbol{e}^T\boldsymbol{e} - 1) = \max \quad (21)$$

By performing the derivative with respect the Euler angle we obtain

$$\begin{aligned} \frac{d\sigma_w^*}{d\Phi} &= -\sin\Phi\ \mathrm{tr}[\boldsymbol{B}] + \sin\Phi\ \boldsymbol{e}^T\boldsymbol{B}\boldsymbol{e} \\ &\quad + \cos\Phi\ \boldsymbol{f}^T\boldsymbol{e} = 0 \quad (22) \end{aligned}$$

and therefore

$$\begin{cases} \sin\Phi = \boldsymbol{f}^T\boldsymbol{e} \\ \cos\Phi = \mathrm{tr}[\boldsymbol{B}] - \boldsymbol{e}^T\boldsymbol{B}\boldsymbol{e}, \end{cases} \quad (23)$$

Let us take the derivative of equation (21) with respect the Euler axis $\boldsymbol{e}$

$$\begin{aligned} \frac{d\sigma_w^*}{d\boldsymbol{e}} &= (1 - \cos\Phi)(\boldsymbol{B}^T + \boldsymbol{B})\boldsymbol{e} \\ &\quad + \sin\Phi\ \boldsymbol{f} - 2\lambda\boldsymbol{e} = 0 \quad (24) \end{aligned}$$

Premultiplying this equation by $\boldsymbol{e}^T$ and keeping in mind the constraint ($\boldsymbol{e}^T\boldsymbol{e} = 1$) the Lagrangian multiplier expression becomes

$$2\lambda = 2(1 - \cos\Phi)\boldsymbol{e}^T\boldsymbol{B}\boldsymbol{e} + \sin\Phi\ \boldsymbol{f}^T\boldsymbol{e} \quad (25)$$

therefore equation (24) may be written as

$$\begin{aligned} \boldsymbol{e} &= \{[2(1 - \cos\Phi)\boldsymbol{e}^T\boldsymbol{B}\boldsymbol{e} + \sin\Phi\boldsymbol{f}^T\boldsymbol{e}]\boldsymbol{I} \\ &\quad + \{(\cos\Phi - 1)(\boldsymbol{B}^T + \boldsymbol{B})\}^{-1} \sin\Phi\boldsymbol{f} \quad (26) \end{aligned}$$

equation (26), along with equation (23), can be used as the recursive equation of an iterative procedure, whose steps are as follows: Euler axis $\boldsymbol{e}_0$ starting point choice, loop on the evaluation of Φ by using equation (23) and computation of the updated Euler axis e_{i+1} by using equation (26). The procedure end is reached when $\cos^{-1}(e_i^T e_{i+1}) <$ tolerance.

NUMERICAL TESTS

All the performed tests are based on attitude data random productions and the average outputs are plotted. These tests are carried out on a 486-PC with *MATLAB* software. Since *EULER*-2 and *TRIAD*-2

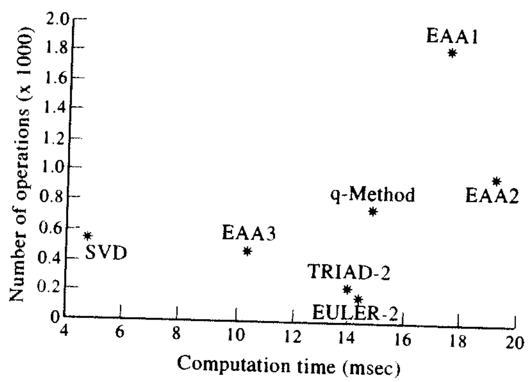

Fig. 4. Speed test comparison (noise < 1°, $n = 2$).

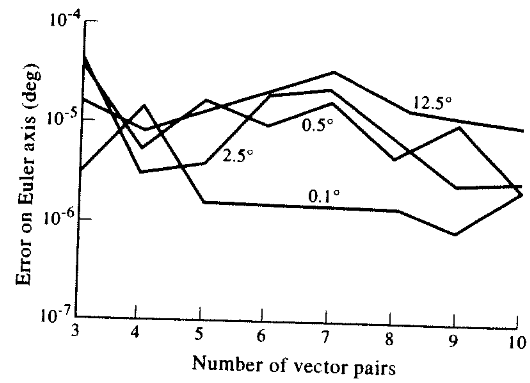

Fig. 6. *EULER-n* accuracy test: Euler axis error.

are optimal and deterministic, they have been tested only in terms of computational speed.

Figure 4 shows, for a random noise less than 1°, the computation time (using the *cputime* function) versus the approximate cumulative number of floating point operations (using the *flops* function), for the algorithms *EULER*-2, *TRIAD*-2, *EAA*-1,2,3 [3], *SVD* [4] and *q-Method* [5]. Owing to lack of time no test comparison has been made with *QUEST* [2].

We outline here that, since the speed index based on the *flops* function is independent of both the hardware and the software used (which is not true for the *cputime* function), it is more reliable.

Tests on *EULER-n* are referring to an angle convergence condition between two consecutive Euler axis iterations of 0.1 degrees. Although the convergence of the method is not demonstrated, it never failed during extensive tests. The iterations number depends on the recursive equation and on the $\boldsymbol{e}_0$ starting point choice. The $\boldsymbol{e}_0$ here suggested is that evaluated by using *EULER*-2 for the best data (those with the greatest weights α_i, α_j).

Figure 5 plots the above mentioned speed indices as function of the number of vector pairs ($n = 3, \ldots, 10$), for six optimal algorithms and for a random noise less than 1°.

The *cputime* differences in performance between *SVD* and the other methods are, however, unreal due to the use of *MATLAB*. In *MATLAB*, the singular value decomposition is highly optimized and precompiled from native C code.

Figures 6 and 7 show the *EULER-n* accuracy with respect the optimal solution, considering four data noise levels (from 0.1 up to 12.5°) and for a convergence angle of 0.1°. Figure 8 plots the averages of the associated number of iterations.

A better $\boldsymbol{e}_0$ choice can be obtained by computing a weighted Euler axis $\boldsymbol{e}_w$, evaluated from all the m Euler axes $\boldsymbol{e}_k$ obtained for the m combinations of unit-vector pair couples (the number of combinations of n objects chosen in pairs is $m = n!/[(n-2)!2!]$). The liability of such computed Euler axes $\boldsymbol{e}_k$ is a function of the associated weights (α_i, α_j). Obviously this relative liability increases with the product $\alpha_i\alpha_j$. Therefore the proposed $\boldsymbol{e}_w$ choice satisfies the following condition

$$\sigma^* = \sum_{k=1}^{m} \beta_k \boldsymbol{e}_k^T \boldsymbol{e}_w - \lambda(\boldsymbol{e}_w^T \boldsymbol{e}_w - 1) = \max \qquad (27)$$

where

$$\beta_k = \frac{\alpha_i\alpha_j}{\Sigma_{ij}\alpha_i\alpha_j} \; (i \neq j) \qquad (28)$$

The solution of equation (27) is

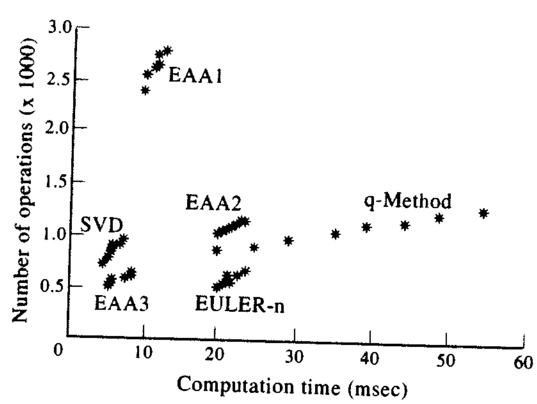

Fig. 5. Speed test comparison (noise < 1°, $n = 3, \ldots, 10$).

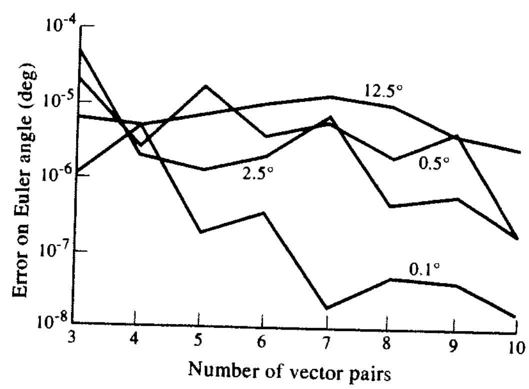

Fig. 7. *EULER-n* accuracy test: Euler angle error.

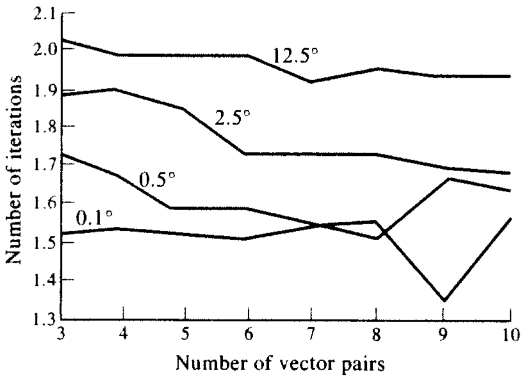

Fig. 8. *EULER-n* accuracy test: iterations number.

$$\mathbf{e}_w = \frac{\sum_k \beta_k \mathbf{e}_k}{|\sum_k \beta_k \mathbf{e}_k|} \tag{29}$$

However, this choice is only apparently advantageous because the iterations gain of using equation (29) does not balance the $\mathbf{e}_w$ involved heavier computational load, especially for a high value of n. It is not necessary, however, to take into account all the m different $\mathbf{e}_k$; limitation to involve only the best two or three $\mathbf{e}_k$ is therefore suggested.

CONCLUSION

In this paper two new optimal attitude determination methods, *EULER*-2 and *EULER-n*, have been developed. *EULER*-2 computes the optimal Euler axis and angle in a deterministic way, and its application is restricted to cases where only two unit-vectors pairs are available. This method is based on the attitude matrix rotation meaning and on a demonstrated co-planarity condition. By using this last condition the extension of the existing *TRIAD* algorithm to the optimal solution, getting *TRIAD*-2, has also been possible. Finally, the *EULER-n* algorithm, which computes the optimal Euler axis and angle by an iterative technique, is developed. The iterative procedure, which starts from the Euler axis evaluated by *EULER*-2, converges to the optimal solution with a precision better than 1/1000 degrees (Figs 6 and 7) and with only one or two iterations (Fig. 8). Numerical tests on computational speed are performed for *EULER*-2, *TRIAD*-2 and *EULER-n* (Figs 4 and 5). These plots show that all these three new algorithms are quite suitable when a fast optimal attitude determination is needed.

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