JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 23, No. 5, September–October 2000

Engineering Notes

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Second Estimator of the Optimal Quaternion

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Introduction

ACCURACY and speed are two very important requirements for attitude estimation algorithms. Usually, the attitude matrix A is estimated using the n directions b_i, observed by the attitude sensors and defined in a body frame, together with the correspondent directions r_i, defined in a reference frame. In terms of the obtained accuracy, the optimality criterion

$$L_W(A) = \frac{1}{2}\sum_{i=1}^{n} \alpha_i \|Ar_i - b_i\|^2 = \min \tag{1}$$

introduced by Wahba[1] in 1965, yields for A an accuracy upper limit as well. Usually, the α_i in Eq. (1) represents the relative weight ($\sum_i \alpha_i = 1$, $i = 1 - n$) associated with the ith observed direction error. The adoption of $\alpha_i = 1/\sigma_i^2$, where σ_i is the standard deviation associated with the b_i direction precision, transforms the minimization of $L_W(A)$ into a maximum likelihood problem.[2,3]

The first direct solution of the Wahba problem was provided in Ref. 4, and many other closed-form solutions were proposed later.[5–13] The breakthrough in the development of the modern fast optimal attitude determination algorithms was given by Paul Davenport, who introduced the q-Method algorithm,[5,6] from which many other methods,[7–13] satisfying Wahba's optimality criterion, were derived. The q-Method demonstrates that the optimal quaternion q_{opt} is the eigenvector associated with the greatest eigenvalue of the 4×4 symmetric matrix K, that is,

$$K q_{\text{opt}} = \lambda_{\max} q_{\text{opt}} \tag{2}$$

where

$$K = \begin{bmatrix} B + B^T - \text{tr}[B] I_{3\times 3} & z \\ z^T & \text{tr}[B] \end{bmatrix} \tag{3}$$

$B = \sum_i \alpha_i b_i r_i^T$ is the attitude profile matrix, $I_{3\times 3}$ the 3×3 unit matrix, and z is the vector $z = \sum_i \alpha_i b_i \times r_i = \{B(2,3) - B(3,2), B(3,1) - B(1,3), B(1,2) - B(2,1)\}^T$. Equation (2) provides the optimal attitude by using very robust existing algorithms, but requires the computation of all of the eigenvalues-eigenvectors. This cumbersome aspect, which slows down the algorithm speed, was solved with the QUEST algorithm,[7] which has shown a way to evaluate only $\lambda_{\max}$ and the associated eigenvector q_{opt}. QUEST evaluates $\lambda_{\max}$ using one/two Newton–Raphson iterations to the K-matrix characteristic equation and, then, q_{opt} is estimated by applying the Cayley–Hamilton theorem together with the Gibbs vector. The use of the Gibbs vector introduces a singularity (for principal angle close to π), which is then solved by the technique of sequential rotations.[7]

The subsequently devised algorithms[8–13] were mainly aimed to search improvement in the computational speed area. In fact, a faster attitude estimation method presents the advantage of allowing the control system to achieve the attitude information at a higher rate. Moreover, because all of these algorithms fully comply with Wahba's optimality criterion, they compute the same attitude matrix; thus, they are equivalent in accuracy and differ from one another only in terms of computational speed.

This Note presents the Second Estimator of the Optimal Quaternion (ESOQ2),[13] which represents a further speed improvement, demonstrated by numerical speed tests, in the area of the optimal attitude estimation algorithms.

Eigenvalue Computation

The $\lambda_{\max}$ computation can be accomplished in many ways that present different characteristics. $\lambda_{\max}$ can be computed in a closed form,[10,13] by using the well-known solution of the quartic algebraic equation associated with the characteristic polynomial of the K matrix

$$\lambda^4 + a\lambda^3 + b\lambda^2 + c\lambda + d = 0 \tag{4}$$

where $a = \text{tr}[K] = 0$, $b = -2(\text{tr}[B]) + \text{tr}[\text{adj}(B + B^T)] - z^T z$, $c = -\text{tr}[\text{adj}(K)]$, and $d = \det(K)$. In particular, when $n = 2$ then $\lambda_{\max}$ can be computed with the simple closed-form expression

$$\lambda_{\max} = \frac{1}{2}\left(\sqrt{2\sqrt{d} - b} + \sqrt{-2\sqrt{d} - b}\right) \tag{5}$$

Alternatively, $\lambda_{\max}$ can be computed by one/two Newton–Raphson iterations

$$\lambda_{i+1} = \lambda_i - \frac{\lambda_i^4 + b\lambda_i^2 + c\lambda_i + d}{4\lambda_i^3 + 2b\lambda_i + c} \tag{6}$$

with $\lambda_0 = \sum_i \alpha_i$ as starting point. The most robust way to compute eigenvalues is to use matrix factorization, such as the QR and the singular value decompositions. However, the gain in robustness (which imply a better numerical stability of the algorithm) is counterbalanced by a heavier required computational load, which slows down the algorithm. Moreover, a high precision method to compute $\lambda_{\max}$ is unnecessary because its value range is nearly negligible about $\lambda_{\max} = \lambda_0$ (Ref. 7). ESOQ2 computes $\lambda_{\max}$ with Eq. (5) when $n = 2$, or with one Newton–Raphson iteration, using Eq. (6). Reference 3 shows that, for a star tracker scenario (all α_i are equal), no iteration, that is, $\lambda_{\max} = \lambda_0$, provides an acceptable accuracy. Anyway, numerical stability is demonstrated by the accuracy tests, which compare the obtained attitude accuracy against that provided using the q-Method approach, for input data slightly in error.

Quaternion Computation

ESOQ1[10] evaluates the optimal quaternion by computing the maximum modulus vector cross product among four cross-product vectors defined in the four-dimensional space. To do this, ESOQ1 implies the computation of seven determinants of 3×3 matrices, whereas ESOQ2 achieves the solution by reducing the size from four to three (by replacing the quaternion with the principal axis and

Received 5 November 1998; revision received 22 October 1999; accepted for publication 17 January 2000.

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angle), which implies the computation of five determinants of 2×2 matrices. As clearly shown later on, the introduced singularity (the principal axis is not defined for a $\Phi = 0$ principal angle) will never occur because the sequential rotation assuring $\pi/2 \le \Phi \le 3\pi/2$ can be identified in advance directly from the input B matrix.

The quaternion can be expressed in terms of the principal axis and angle[14]

$$q^T = \{e^T \sin(\Phi/2),\ \cos(\Phi/2)\} \tag{7}$$

By this substitution Eq. (2) can be split as follows:

$$\begin{aligned} &Se\sin(\Phi/2) + z\cos(\Phi/2) = 0 \\ &z^T e\sin(\Phi/2) + (\mathrm{tr}[B] - \lambda_{\max})\cos(\Phi/2) = 0 \end{aligned} \tag{8}$$

where $S = B + B^T - (\mathrm{tr}[B] + \lambda_{\max})I_{3\times 3}$. It is possible to eliminate the principal angle Φ from Eq. (8); thus the key equation for the optimal principal axis

$$\left[(\mathrm{tr}[B] - \lambda_{\max})S - zz^T\right]e = Me = 0 \tag{9}$$

is obtained, where the 3×3 symmetric matrix M is introduced. Equation (9) states that all of the row/column vectors of M are perpendicular to e. Therefore, the optimal principal axis can easily be evaluated by a simple vector cross product between two rows/columns of matrix M. However, this way to compute e entails the problem of which vectors should be selected for computing it with the highest robustness. To this end, let M be as

$$M = M^T = [m_1 \quad m_2 \quad m_3] = \begin{bmatrix} m_a & m_x & m_y \\ m_x & m_b & m_z \\ m_y & m_z & m_c \end{bmatrix} \tag{10}$$

then, three different choices will be available:

$$\begin{aligned} e_1 &= m_2 \times m_3 \\ &= \left\{m_b m_c - m_z^2,\ m_y m_z - m_x m_c,\ m_x m_z - m_y m_b\right\}^T \\ e_2 &= m_3 \times m_1 \\ &= \left\{m_y m_z - m_x m_c,\ m_a m_c - m_y^2,\ m_x m_y - m_z m_a\right\}^T \\ e_3 &= m_1 \times m_2 \\ &= \left\{m_x m_z - m_y m_b,\ m_x m_y - m_z m_a,\ m_a m_b - m_x^2\right\}^T \end{aligned} \tag{11}$$

All of the e_i provided in Eq. (11) are parallel, and, therefore, they differ only in modulus. To evaluate the optimal principal axis with maximum robustness, it is necessary to identify the e_i with greatest modulus. To do that, Eq. (11) allows us to write $e_i(k) = e_k(i)$, while the condition of parallel e_k allows us to write $|e_k(j)| = \xi_{ki}|e_i(j)|$, where $\xi_{ki} = \|e_k\|/\|e_i\|$, and $j = 1$–3. Therefore, from the preceding it turns out $|e_k(k)| = \xi_{ki}|e_i(k)| = \xi_{ki}|e_k(i)| = \xi_{ki}^2|e_i(i)|$, which demonstrates that the farthest-from-zero and the closest-to-zero elements $e_k(i)$ have $k = i$. Let k be the index associated with the farthest element, that is, $e_k(k)$; then the highest-modulus vector cross product is e_k, as provided by Eq. (11). The second equation of Eq. (8) allows us to write

$$z^T e = h\cos(\Phi/2), \qquad (\lambda_{\max} - \mathrm{tr}[B]) = h\sin(\Phi/2) \tag{12}$$

where h is an unknown constant. Thus, the direction of the optimal quaternion is obtained by normalizing

$$q = \begin{Bmatrix} (\lambda_{\max} - \mathrm{tr}[B])e_k \\ z^T e_k \end{Bmatrix} \tag{13}$$

that is, $q_{\mathrm{opt}} = q/\sqrt{(q^T q)}$.

When approaching the singularity condition, the optimal principal axis, evaluated as a cross product, would be affected by an error, which increases as $\Phi \to 0$. This error will affect the first three elements of the quaternion, elements that are also multiplied by $\sin(\Phi/2)$, which is actually small. This multiplication, therefore, represents a compensation effect analyzed in Ref. 13. However, the computation of the principal axis by means of a vector cross product fails in two limit cases for the M row vectors: 1) when they become parallel, which corresponds to the intrinsic unsolvable case of parallel observed vectors (case that cannot be solved by any attitude estimation algorithm), and 2) when they become zero, which occurs when $\Phi \to 0$. The latter case, which represents the singularity condition introduced by the method, is easily avoided as explained in the next section.

Table 1 Sequential rotation table

Rotation matrix	$\bar{B}$ matrix ($B = \bar{B}R$)	$\bar{A}$ matrix ($A = \bar{A}R$)	q^T quaternion
$R(c_1, \pi)$	$\bar{B}_1 = [b_1, -b_2, -b_3]$	$\bar{A}_1 = [a_1, -a_2, -a_3]$	$\{\bar{q}_4, -\bar{q}_3, \bar{q}_2, -\bar{q}_1\}^T$
$R(c_2, \pi)$	$\bar{B}_2 = [-b_1, b_2, -b_3]$	$\bar{A}_2 = [-a_1, a_2, -a_3]$	$\{\bar{q}_3, \bar{q}_4, -\bar{q}_1, -\bar{q}_2\}^T$
$R(c_3, \pi)$	$\bar{B}_3 = [-b_1, -b_2, b_3]$	$\bar{A}_3 = [-a_1, -a_2, a_3]$	$\{-\bar{q}_2, \bar{q}_1, \bar{q}_4, -\bar{q}_3\}^T$

How to Avoid Singularity

The Shuster's sequential rotation technique[7] states that if the n unit vector pairs (b_i, r_i) imply the attitude matrix A, then n pairs $(b_i, \bar{r}_i)$, where $\bar{r}_i = Rr_i$ and R is any rotational matrix, imply the attitude matrix $\bar{A} = AR^T$. Thus, if the (b_i, r_i) data set implies a singularity for the application method, the $(b_i, \bar{r}_i)$ data set should not, in general, necessarily imply a singularity, too. Hence, the method evaluates the attitude $\bar{A}$ by using the rotated unit vectors $\bar{r}_i$ (in place of the r_i) and then computes the searched optimal attitude matrix as $A = \bar{A}R$. Particular attention is given to those matrices R, which rotate about one of the coordinate axes c_1, c_2, and c_3 by a π angle. Let these matrices be indicated as $R(c_1, \pi)$, $R(c_2, \pi)$, and $R(c_3, \pi)$. With respect to others, the use of these matrices does not involve any extra computational loads, but sign changes and cross displacements, only. Table 1 shows the relationships between the attitude profile matrix B, the attitude A, and the quaternion q, with the associated rotated quantities $\bar{B}$, $\bar{A}$, and $\bar{q}$.

The important property $\mathrm{tr}[B] \ge \cos\Phi$ (which is not demonstrated here) allows us to establish if a sequential rotation is needed.[3] In fact, when $\mathrm{tr}[B] < \cos(\Phi_{\min})$, where $\Phi_{\min}$ is the minimum acceptable principal angle (which assures to be far enough from singularity), then no sequential rotation is needed. When $\mathrm{tr}[B] \ge \cos(\Phi_{\min})$, then ESOQ2 checks if a π sequential rotation about the c_1 coordinate axis implies a principal angle $\bar{\Phi} > \Phi_{\min}$ by checking if $\mathrm{tr}[\bar{B}_1] = B(1,1) - B(2,2) - B(3,3) < \cos\Phi_{\min}$. Similarly, if even this check fails, then it is possible to investigate, in sequence, if the π sequential rotation about the c_2 axis is such that $\mathrm{tr}[\bar{B}_2] = B(2,2) - B(1,1) - B(3,3) < \cos\Phi_{\min}$ and also if the π sequential rotation should be accomplished about the c_3 coordinate axis by checking if $\mathrm{tr}[\bar{B}_3] = B(3,3) - B(1,1) - B(2,2) < \cos\Phi_{\min}$. It is not possible that all of the sequential rotation checks fail for $\Phi_{\min} \le \pi/2$. The preceding procedure allows us to ensure the condition $\pi/2 \le \bar{\Phi} \le 3\pi/2$, just by setting $\Phi_{\min} = \pi/2$ (at least, one of the elements $\{\mathrm{tr}[\bar{B}_1]\ \ \mathrm{tr}[\bar{B}_2]\ \ \mathrm{tr}[\bar{B}_3]\ \ \mathrm{tr}[B]\}$ must be negative).[3] To ensure $\pi/2 \le \bar{\Phi} \le 3\pi/2$, if the negative element is the kth ($k = 1$–3), then the π sequential rotation about the c_k axis is needed; otherwise, no rotation can be accomplished.

Unlike QUEST,[7] for which the sequential rotations technique could even be applied three times, ESOQ2 may need it only one time. Moreover, because the condition $\pi/2 \le \bar{\Phi} \le 3\pi/2$ is satisfied, the ESOQ2 algorithm also works near the most robust position ($\Phi = \pi$) to estimate the principal axis.

Speed Tests

Reference 10 has already demonstrated that ESOQ1 was the fastest existing optimal attitude estimation algorithm. Figure 1 shows an overall speed test comparison between ESOQ2 and

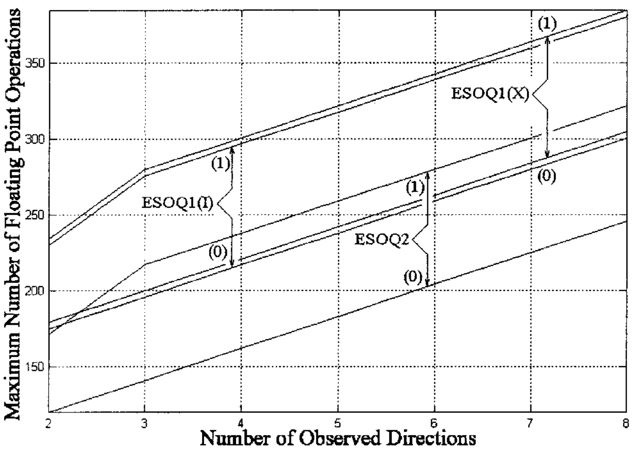

Fig. 1 Speed tests results.

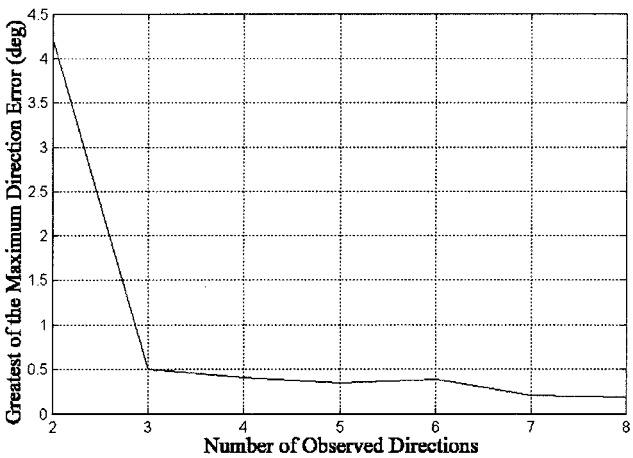

Fig. 2 Accuracy tests results.

ESOQ1, using either the four-dimensional vector cross product ESOQ1(X) or the 3×3 matrix invertion ESOQ1(I), and with one or zero Newton–Raphson iterations to compute $\lambda_{\max}$. For the tests performed in Fig. 1, MATLAB® software has been used. The data set, which consists of the n data (α_i, b_i, r_i) as well as the attitude matrix T, is $N = 1000$ times randomly produced, and the range of vector pairs n goes from 2 to 8. The sensors' precision are here simulated with uniform error distribution by rotating the unit vectors Tr_i about a random axis orthogonal to Tr_i by a random angle b_i linearly correlated to the weight α_i and not greater than 0.5 deg. The index used to evaluate the algorithm computational speed is the MATLAB[15] function, which allows the evaluation of the approximate cumulative number of floating point operations. This index, unlike the consumed computational time, is independent of both the software and the hardware used.

Figure 1 shows ESOQ2 with a constant speed gain with respect to ESOQ1, for any number of the observed vectors considered. This gain ranges from 55–60 up to 60–65 floating point operations, when one or zero iteration is used to evaluate $\lambda_{\max}$, respectively. This result makes ESOQ2 the most suitable optimal attitude estimation algorithm when the highest attitude estimation speed is required.

Accuracy Tests

Let T and A be the true and the estimated attitude matrices. The error ε, associated with an observed direction b, is the angle between $T^T b$ and $A^T b$. This angle, which is a function of the direction b, has a spatial distribution with a maximum value provided by $\cos \varepsilon_{\max} = (\mathrm{tr}[T A^T] - 1)/2$. With the same attitude data set used for the speed tests, Fig. 2 plots the greatest values of $\varepsilon_{\max}$ obtained in $N = 1000$ tests for each number of the observed vectors, and for both the estimated matrices provided by q-Method and ESOQ2. Note that q-Method computes $\lambda_{\max}$ with a robust matrix factorization method, and ESOQ2 just uses $\lambda_{\max} = \sum_i \alpha_i = 1$ (no iterations). The differences between the two obtained curves, which practically overlap each other, is actually very small (about 10^{-3} deg).

Conclusions

The ESOQ2 algorithm for a fast optimal estimation of the spacecraft attitude is presented here. ESOQ2 starts from the q-Method solution equation and, therefore, requires the computation of the maximum eigenvalue $\lambda_{\max}$ of a 4×4 symmetric matrix K. $\lambda_{\max}$ is evaluated with the closed-form solution if the observed directions are only two while, otherwise, with one Newton–Raphson iteration applied to the characteristic equation of matrix K.

ESOQ2 computes the optimal quaternion through the evaluation of the optimal principal axis. The introduced singularity is fully avoided using only one sequential rotation. It is demonstrated that the principal axis is the eigenvector associated with the zero eigenvalue of the 3×3 symmetric matrix M. This allows its computation by means of a vector cross product between two row/column vectors of matrix M. Then, the optimal quaternion is straightforwardly evaluated.

Numerical tests show ESOQ2 as the fastest among the nonsingular and optimal attitude estimation algorithms. The robustness of the method is also demonstrated by numerical accuracy tests. This makes it the most suitable attitude estimation algorithm when fast spacecraft attitude estimation is required.

Acknowledgments

This work was supported by Italian Space Agency Under Contract ARS 98-79. I should like to thank Bruno Bernabei for the tedious task of reading through the manuscript, F. Landis Markley for useful discussions on the attitude determination problem, and Carlo Arduini. This Note was extracted from Paper AAS 97-167 of the 7th Annual AAS/AIAA Space Flight Mechanics Meeting, Huntsville, Alabama, 10–12 February 1997.

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Improving Time-Optimal Maneuvers of Two-Link Robotic Manipulators

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I. Introduction

THIS Note discusses in detail numerical results of a parametric study for rest-to-rest time-optimal maneuvers of rigid two-link robotics manipulators. The problem of minimizing the time of planar maneuvers in terms of control and structure was considered parametrically. First, the time-optimal control strategy was applied. This strategy led to a bang-bang control in which the motors operated with the maximum torques changing directions at the switch time. The solutions were obtained by directly using Pontryagin's minimum principle (PMP). The analysis was then repeated for different lengths of particular links and for different torques applied at particular joints. The total length of the manipulator and the resultant torques generated by the shoulder and elbow motors were kept constant. For the numerical calculations, the set of data characterizing the IBM 7535 B 04 robot discussed in Refs. 1 and 2 was adapted.

II. Time-Optimal Control of Two-Link Robotic Manipulators

The equations derived from PMP and the corresponding boundary conditions form a two-point boundary value problem (TPBVP). Here we present the solutions generated by a numerical procedure that combines the forward-backward method with the shooting method to directly solve the TPBVP.[3] The procedure that is capable of determining the states, the costates, and the switching functions with a high numerical accuracy was discussed in more detail in Ref. 4. The procedure was used by the authors in Ref. 5 examine the effects of orientation of the plane of motion on the time-optimal maneuvers in the gravitational field. That work is extended here to include the effects of the links and the torque ratios.

Maneuvers of a two-link robotics manipulator are considered in plane y, z as shown in Fig. 1. Mass moments of inertia of the links with respect to their centers of mass, located at l_{c1} and l_{c2}, respectively, are I_1 and I_2. The states are $x_1 = \varphi_1$, $x_2 = \dot{\varphi}_1$, $x_3 = \varphi_2$, and $x_4 = \dot{\varphi}_2$. The manipulator is driven by motors installed at the shoulder and elbow joints and generating torques u_1 and u_2, respectively.

In terms of the states x_i, $i = 1, \ldots, 4$, the equation of motion of the manipulator can be obtained in the form

$$\dot{x}(t) = \boldsymbol{A}(x) + C(x)u(t) \tag{1}$$

where $\boldsymbol{A}$ and C are the vector and the matrix of nonlinear functions of states (see Ref. 4), and $\boldsymbol{u}$ is the vector of controls, represented here by torques $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$. The control torques are bounded as

$$U_i^- \leq \boldsymbol{u}_i(t) \leq U_i^+ \tag{2}$$

For the time-optimal control problem, the state departing from the initial conditions, $x(0) = x_0$, must reach the final conditions, $x(t_f) = x_f$, in a minimum time that is $t_f \to \min$. After introducing the costates $p(t)$ and applying Pontryagin's Minimum Principle, the optimal solution must satisfy the necessary conditions

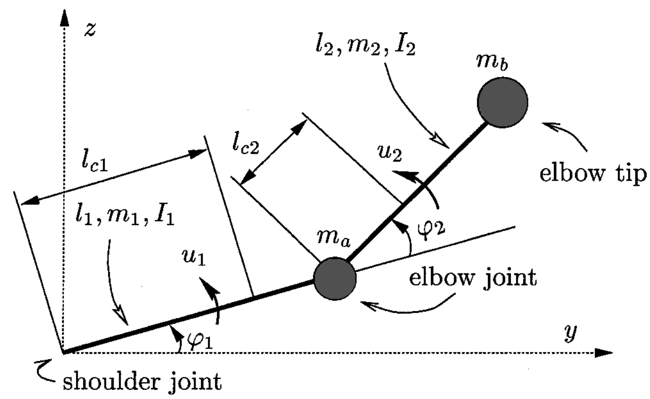

Fig. 1 Planar two-link robotics manipulator.

$$\dot{x} = \frac{\partial H}{\partial p} \qquad \dot{p} = -\frac{\partial H}{\partial x} \tag{3}$$

where the Hamiltonian $H(x, u, p) = 1 + p^T[A(x) + C(x)u(t)] \xrightarrow{u} \min$. The control torques have the form of a bang-bang control

$$u_i = \begin{bmatrix} U_i^+ & \text{for} & G_i < 0 \\ U_i^- & \text{for} & G_i > 0 \end{bmatrix} \tag{4}$$

where $G_i = p_j C_{ji}$ is the switch function corresponding to the control u_i, and C_{ji} are the components of ith column of matrix C. For the time-optimal trajectory, the Hamiltonian must satisfy the condition that $H(x, u, p) = 0$ for all t.

The solution of four state and four costate equations must satisfy eight initial and final conditions imposed on the state only. This TPBVP is solved numerically for assumed values of x_0, x_f, U_i^-, and U_i^+ using the procedure discussed in Refs. 3 and 4. Any solution that meets PMP for the manipulator as defined earlier is referred to as time-optimal. Here we concentrate on discussing some of these solutions.

III. Simulation Results

In the analysis presented, it is assumed that the links are made of cylindrical uniform bars of diameters d_i for which $I_i = m_i(l_i^2/12 + d_i^2/16)$. Additionally, for the purpose of this analysis, it is assumed that the total length of the manipulator $L_t = l_1 + l_2$ and the total torque $U_t = U_1 + U_2$ are constant. The calculations are performed for different length and different torque ratios defined by

$$R_L = l_1/l_2 \qquad R_U = U_1/U_2 \tag{5}$$

If $L_t = 0.65$ m, $R_L = 1.6$ and $U_t = 34$ Nm, $R_U = 2.7778$, and for $d_1 = d_2 = 0.10987$ m the following parameters are identical to those given in Refs. 1 and 2 for the IBM 7535 B 04 robot:

$$l_1 = 2l_{c1} = 0.4\ \text{m} \qquad U_1^{\mp} = \mp 25\ \text{Nm} \qquad I_1 = 0.4167393\ \text{kg}\cdot\text{m}^2$$
$$l_2 = 2l_{c2} = 0.25\ \text{m} \qquad U_2^{\mp} = \mp 9\ \text{Nm} \qquad I_2 = 0.1102435\ \text{kg}\cdot\text{m}^2$$
$$m_1 = 29.58\ \text{kg} \qquad m_2 = 18.49\ \text{kg} \tag{6}$$

In terms of the states, the initial and the final conditions for the rest-to-rest maneuvers from straight-to-straight configurations are given as $x(0) = [0.0\ \ 0.0\ \ 0.0\ \ 0.0]^T$, $x(t_f) = [\varphi_{1_f}\ \ 0.0\ \ 0.0\ \ 0.0]^T$, where φ_{1_f} is final maneuver angle.

A. Effects of the Length Ratio

Theoretically, the manipulator can access any point within the circle of radius L_t only if $R_L = 1$. Therefore, in order to improve accessibility, the length ratio for the robot analyzed in the previous

Received 9 April 1999; revision received 15 January 2000; accepted for publication 24 February 2000.

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